1. A time series $Z_t$ can be described by an ARMA(1,1) model. Assume that a realization $Z_1, Z_2, \ldots Z_{100}$ is available to estimate all of the parameters in this model and to compute forecasts and that a forecast is needed for $Z_{102}$, using $Z_{100}$ as the forecast origin.

(a) Give an expression for $Z_{102}$, based on the ARMA(1,1) model.

(b) Give an expression that can be used to compute $\hat{Z}_{100}(2)$, the forecast for $Z_{102}$.

(c) Give an expression for $e_{100}(2)$, the forecast error in $\hat{Z}_{100}(2)$.

(d) Give an expression for the variance of $e_{100}(2)$.

(e) Give an expression for a 95% prediction interval for $Z_{102}$. 
2. Consider the seasonal time series model for sales (in dollars per month) defined by

\[ W_t = (1 - \theta_1 B)(1 - \Theta_1 B^{12})a_t \]

with the differencing scheme

\[ W_t = (1 - B)(1 - B^{12})Z_t. \]

(a) Write down the unscrambled equation for \( Z_t \).

(b) Is \( Z_t \) stationary or not? Explain briefly.

(c) Derive the variance \( \text{Var}(W_t) \).

(d) Derive an expression for the autocorrelation \( \rho_{13} \) for \( W_t \).
3. Consider the ARMA model

\[(1 - \phi_2 B^2)Z_t = a_t\]

(a) Note that the model can also be expressed as \(Z_t = \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \ldots\). Derive expressions for \(\psi_1\), \(\psi_2\), and \(\psi_3\) for this model.

(b) What is the condition that needs to be satisfied for this model to be stationary?

(c) Derive an expression for \(\text{Cov}(Z_t, a_{t-2})\) for this model.

(d) Is this model invertible? Explain why or why not.

4. Experience has shown us that doing a transformation on the response variable can have an important effect on the resulting forecasts. The range mean plot provides a preliminary indication of the need for a transformation. There are, however, better tools to help make the final decision on what transformation (if any) to use. Explain.
5. A company manufactures a product, but does not use television advertising for marketing. A marketing analyst argues that sales of their product \((y_t)\) is sales in week \(t\) depends on past and present television advertising levels for a competing product \((x_t)\) is the competitor’s expenditures for advertising in week \(t\). Historically, the company’s sales have been stationary (no important seasonality or trend). Weekly time series data are available for both \(x\) and \(y\). The marketing managers are interested in developing a time series model that will provide short term forecasts and prediction intervals for future values of \(y\) so that steps can be taken to avoid problems of under stock or over stock. An analyst has identified the transfer function model as

\[
y_t = \nu_3 x_{t-3} + \nu_4 x_{t-4} + (1 - \phi_1 B)^{-1}(1 - \theta_1 B)a_t
\]  

The model for \(x\) has been identified as an AR(2).

(a) Write down the unscrambled model for \(y\) (only finite lags allowed).

(b) List the steps that you would use to identify the form of the transfer function model.

1. 
2. 
3. 
4. 
5. 

(c) Model (1) implies no feedback. Is it possible that there is a feedback relationship between \(x\) and \(y\) in the actual problem? Explain why or why not and explain how the feedback could be detected if it exists.
6. A sample realization can be described by the model

\[ Z_t = \theta_0 + \phi_1 Z_{t-1} + a_t. \]

An analyst decides to try the differencing scheme

\[ W_t = (1 - B)^2 Z_t \]

(a) Write down the model equation for the derived time series \( W_t \). Note that \( Z \) needs to be eliminated from the model.

(b) Is \( W_t \) stationary? Why or why not?

(c) Is \( W_t \) invertible? Why or why not?