Due 9/18/12

1. Consider the operator \( \mathcal{L} \) defined on doubly infinite vectors of real numbers by the relationship

\[
Z = \mathcal{L}Y \implies z_t = y_{-t}
\]

(\( \mathcal{L} \) reverses time).

(a) Is \( \mathcal{L} \) a linear operator? If so, what "matrix" is associated with it? If not, argue carefully that it is not.

(b) Is \( \mathcal{L} \) invertible? If so, what is an inverse operator for it? If not, argue carefully that it is not.

(c) Is \( \mathcal{L} \) "time invariant"? If so, specify the impulse response function. If not, argue carefully that it is not.

2. Consider the operator \( \mathcal{I} - \phi_1 B - \phi_2 B^2 \). Suppose that the constants \( \phi_1 \) and \( \phi_2 \) are such that this operator has an inverse that is a causal time-invariant linear filter with impulse response function given by the absolutely summable values \( \psi_0, \psi_1, \psi_2, \ldots \).

(a) Find the values of \( \psi_0, \psi_1, \) and \( \psi_2 \) and a recursion for \( \psi_t \) in terms of \( \psi_{t-1} \) and \( \psi_{t-2} \) for \( t \geq 2 \).

(b) Suppose in addition that both \( \phi_1 \) and \( \phi_2 \) are positive. Argue that their sum cannot exceed 1. Hint: You should be able to use the recursion to find \( \sum_{s=0}^{\infty} \psi_s \) in terms of \( \phi_1 \) and \( \phi_2 \).

3. Consider the argument in the course outline to the effect that under appropriate conditions on coefficients \( \phi_1, \phi_2, \ldots, \phi_p \), the operator \( \mathcal{I} - \Phi(B) = \mathcal{I} - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p \) could have an inverse representable as

\[
\sum_{s=0}^{\infty} (\Phi(B))^s
\]

A crude condition is this. Suppose that \( \sum_{s=1}^{p} |\phi_s| < 1 \).

Let \( \|Y\|_\infty \) be a measure of "size" of \( Y \) defined as

\[
\sup_t |y_t|
\]

(so that each \( |y_t| \leq \|Y\|_\infty \)). Argue that for any \( Y \) in \( L_\infty(\mathbb{Z}) \), the part of \( \mathbb{R}^\infty \) consisting of \( Y \) with finite size, for \( \mathcal{L}_k = \sum_{s=0}^{k} (\Phi(B))^s \)

\[
\|Y - \mathcal{L}_k (\mathcal{I} - \Phi(B)) Y\|_\infty \to 0
\]

and hence that the representation of \( (\mathcal{I} - \Phi(B)) \) is sensible. (You just need to argue that \( \| (\Phi(B))^{k+1} Y \|_\infty \) must go to 0 as \( k \) increases.)

4. Argue carefully that the composition of the two operators \( \sum_{k=0}^{l} \alpha_k B^k \) and \( \sum_{j=0}^{m} \beta_j B^j \) is the operator

\[
\sum_{k=0}^{l} \sum_{j=0}^{m} \alpha_k \beta_j B^{k+j}
\]

(so "powers" of the backshift, i.e. successive compositions of it, operate formally as do ordinary algebraic operations on variables).

5. Argue carefully that ordinary differencing and seasonal differencing at \( s \) periods commute, i.e. there is no difference between the operators

\( \mathcal{D}D_s \) and \( D_s \mathcal{D} \)

(For sake of concreteness, consider \( s = 4 \) only. What are these two operators in terms of the polynomials in the backshift operator?)
6. Consider the "pure polynomial trend" series \( L \) and \( Q \) where

\[
l_t = a + bt \quad \text{and} \quad q_t = a + bt + ct^2
\]

What are
(a) \( D_L \) and \( D_Q \)?
(b) \( D^2_L \) and \( D^2_Q \)?
(c) \( D^3_L \) and \( D^3_Q \)?
(d) \( D_4L \) and \( D_4Q \)?

7. Consider the "linear trend plus sinusoidal seasonal effects" series \( L \) defined by

\[
l_t = a + bt + \sin\left(\frac{t\pi}{2}\right)
\]

What are
(a) \( D_L \)?
(b) \( D^2_L \)?
(c) \( D^3_L \)?
(d) \( D_4L \)?
(e) \( D_4DL \)?

8. Using \texttt{JMP} (or some other vehicle if you wish) simulate and plot (versus \( t \), connecting successive plotted points with line segments) values of times series \( y_1 \) through \( y_{100} \) based on iid standard normal \( \epsilon_t \)'s and the models indicated below.

(a) White noise.
(b) The four AR(1) models for \( \phi_1 = \pm 0.3 \) and \( \phi_1 = \pm 0.7 \).
(c) The four MA(1) models for \( \theta_1 = \pm 0.3 \) and \( \theta_1 = \pm 0.7 \).
(d) The four MA(2) models for \( \theta_1 = \pm 0.3 \) and \( \theta_2 = \pm 0.3 \).
(e) The four ARMA(1,1) models for \( \phi_1 = \pm 0.3 \) and \( \theta_2 = \pm 0.3 \).

9. Plot the autocorrelation functions for all of the models in Problem 8. (Plot them as properly oriented spikes at the integers.)

\textbf{Due 10/12/12} (You may do this problem in groups of 2-4, turning in a single write-up for the whole group ... BUT, if you do so, every person’s name on the paper is an implicit statement that the person was a full participant in the development of the solution.

10. There is a sample dataset in \texttt{JMP} named "\texttt{GNP.jmp}" that will be used in this extended exercise/data analysis. The plan here is to find both "ARIMA" and "transfer function" models for this scenario. The basic series available in that dataset are

\[
y_t = \text{gross national product (GNP) at quarter } t
\]
\[
x_{1t} = \text{personal consumption expenditures (PCE) at quarter } t
\]
\[
x_{2t} = \text{gross private domestic investment (GPDI) at quarter } t, \text{ and}
\]
\[
x_{3t} = \text{net exports of goods and services (NE) at quarter } t
\]

and we’ll take as our ultimate goal the forecasting of \( y_t \).
(a) To begin, create new columns in your data table for 3 new versions of each of the $y, x_1$, and $x_2$ series by taking (natural) log, square root, and fourth root of the original series. Because some values of $x_3$ are negative, you won't be able to do this transforming with $x_3$ directly. You might do something similar to the square root and fourth root by using the transforms

$$f(x) = \text{sgn}(x)|x|^{1/2} \quad \text{and} \quad g(x) = \text{sgn}(x)|x|^{1/4}$$

for $\text{sgn}(x) = I[x > 0] - I[x < 0]$. A possibility for a modification of the logarithm would be the transform

$$h(x) = I[|x| \geq 2]\text{sgn}(x)\ln(|x|) + I[|x| < 2] \frac{x \ln 2}{2}$$

(b) In the JMP™ Time Series drop-down menu, there is a "Difference" option. Use it for all of the series $y_t, \ln(y_t), (y_t)^{1/2}$, and $(y_t)^{1/4}$, in consideration of various differencing possibilities of the form $D^dD^p_t$, looking for one (or possibly a couple) combination(s) of "low order" differencing scheme and transform for which second order stationarity looks like a "not terrible" model assumption.

(c) For the possibility (or possibilities) that you identify in (b), identify sensible ARIMA/SARIMA models for the transformed variable. Consider model parsimony (small $p, q, P, Q$), apparent "iidness" of residuals, statistical significance of estimated model parameters, good "fits" as judged by the various potential criteria provided in the JMP™ "Model Comparison" report(s), etc.

(d) Save 95% prediction limits for your models from (c) for $s = 12$ periods beyond the data in hand. "Untransform" these to make prediction limits for 12 future values of $y_t$. Plot these versus $t$ on a single set of axes (using different symbols for each set of limits) for purposes of comparing the limits you create. How much difference do you find across these sets? Do you have plausible "explanations" for any differences that you see?

(e) Choose your very favorite model from (c), and do the following. In succession, double the values of the modeled series at $t = 25, 50, 75, 100, 125$ introducing outliers/errors into the data set. (My intention here is that you have 5 different contaminated data sets, each differing from the original data set at only 1 time point.) Refit your favorite model 5 times and compare fitted parameters to the original values of those parameters. How much change is there in these? Also, compare forecasts $s = 4, 8, 12$ periods into the future to the original forecasts.

(f) Again for your very favorite model from (c), do the following. Introduce a permanent step change of size +1500 in $y_t$ into the raw data successively at times $t = 25, 50, 75, 100, 125$. (My intention here is that you have 5 different contaminated data sets, each with a single step change.) Refit your favorite model 5 times and compare fitted parameters to the original values of those parameters. How much change is there in these? Also, compare forecasts $s = 4, 8, 12$ periods into the future to the original forecasts.

(g) Compare fits for parts (e) and (f) with the changes at time $t = 100$, ignoring the changes in the data to what you get employing a "pulse at time $t = 100" covariate in the first case and a "level shift at time $t = 100" covariate in the second. (In parts (e) and (f) you ignored the interventions in the fitting. Here model the interventions.) As far as I can tell, you are going to have to make up your own $x$ series for these. Be sure to difference these the same way you difference the (possibly transformed) $y$ series. (You'll need to use the JMP™ "Transfer Function" facility.)

(h) Now consider modeling $y_t$ or your favorite transform of it using the $x$ series as predictors. (You probably want to difference $x$'s in whatever way you difference the GNP variable.) Find and compare several plausible "transfer function" models here. Again make use of the JMP™ "Model Comparison" report(s), etc. as in (c). As a means of trying to identify plausible orders for backshift polynomial operators to apply to the predictor series, you can save BOTH residuals from an ARIMA model for $y$ (or transformed $y$) and differenced versions of the predictor series. Then applying the JMP™ Time Series analysis to these series, choosing "Cross Correlation" from the Time Series drop-down menu, you can look for identities of series and lags at which there seems to be some correlation with the residuals for the GNP variable. As you look for plausible models, keep in mind that want model simplicity, good fit, good predictions, etc. For
the model you fit to be of much practical use, you almost surely want to use only positive lags for the predictors in modeling.

(i) How does your favorite transfer function model from (h) compare to your favorite ARIMA model from (c) in terms of its apparent fit to the data?

(j) See what JMP will do about giving you access to forecasts $s = 4, 8, 12$ periods into the future to the original forecasts. (If it will allow you save prediction limits, repeat (d). If not, but you can read prediction limits off a screen using the JMP "Crosshair Tool," do so. If JMP will allow you to manipulate values for future ($t > 126$) values of predictor series, determine which ones impact forecasts at times $t = 126 + s$ for $s = 1, 2, 3, 4$ for your favorite model from (h). Is what you find consistent with the form of your transfer function?)

(k) Repeat (e) and (f) using your favorite model from (h).