(a). $OC = P(a|\theta) = \sum_{(n,x_n) \in A} P(X_n = (n, x_n))$

$$= 3 \cdot \left\{ p^5(1-p)^3 + p^3(1-p)^5 \right\} + 5.3 \cdot \left\{ p(1-p)^4 + p^5(1-p)^2 \right\} + 7.9 \cdot \left\{ p^2(1-p)^5 + p^5(1-p)^2 \right\}$$

$$+ 9.27 \cdot \left\{ p^5(1-p)^4 + p^7(1-p)^2 \right\} + 10.81 \cdot \left\{ p^7(1-p)^4 + 11.162 \cdot \left\{ p^5(1-p)^4 + p^7(1-p)^2 \right\} \right\}$

$$= 3 \cdot \left\{ p^5(1-p)^3 + p^3(1-p)^5 \right\} + 5.3 \cdot \left\{ p(1-p)^4 + p^5(1-p)^2 \right\} + 7.9 \cdot \left\{ p^2(1-p)^5 + p^5(1-p)^2 \right\}$$

$$+ 9.27 \cdot \left\{ p^5(1-p)^4 + p^7(1-p)^2 \right\} + 10.81 \cdot \left\{ p^7(1-p)^4 + 11.162 \cdot \left\{ p^5(1-p)^4 + p^7(1-p)^2 \right\} \right\}$

(b). $N = 100$

$$AOQ = \sum_{(n,x_n) \in A} (1 - \frac{n}{N}) \cdot P_r(ending \ at \ (n,x_n))$$

$$= \left(1 - \frac{3}{100} \right) \cdot p^3(1-p)^3 + \left(1 - \frac{5}{100} \right) \cdot p \cdot 3p(1-p)^4 + \left(1 - \frac{7}{100} \right) \cdot p \cdot p^5(1-p)^2 + \left(1 - \frac{9}{100} \right) \cdot p \cdot 7p(1-p)^6$$

$$+ \left(1 - \frac{10}{100} \right) \cdot 8p(1-p)^6 + \left(1 - \frac{12}{100} \right) \cdot p \cdot 16p^5(1-p)^6$$

$$ATI = N \cdot (1 - Pa) + \sum_{(n,x_n) \in A} n \cdot P_r(ending \ at \ (n,x_n))$$

$$= 100 \cdot (1 - Pa) + 3p^5(1-p)^3 + 5.3p(1-p)^4 + 7.9p^5(1-p)^2 + 9.27p(1-p)^6$$

$$+ 10.81p^7(1-p)^4 + 11.162p^5(1-p)^6$$
(a) Find single limit variables plan.

\[ L = 1.000 \quad \sigma = 0.015 \quad p_1 = 0.03 \quad Pa_1 = 1.95 \]

\[ p_2 = 0.10 \quad Pa_2 = 1.10 \]

\[ n \approx \left( \frac{\sigma^2 (p_1) - \sigma^2 (p_2)}{\sigma^2 (1-p_1) - \sigma^2 (1-p_2)} \right)^2 = \left( \frac{0.015^2 (1 - 0.03) - 0.015^2 (1 - 0.10)}{0.015^2 (1 - 0.03) - 0.015^2 (1 - 0.10)} \right)^2 = \frac{1.6449 - (-1.2816)}{1.6449 - (-1.2816)} \approx 23.85 \approx 24 \]

\[ \Delta_1 \approx \sigma \left( \frac{\sigma (p_1) \sigma (1-p_2) - \sigma (p_2) \sigma (1-p_1)}{\sigma^2 (p_1) - \sigma^2 (p_2)} \right) = 0.015 \left( \frac{(1.6449)(1.2816) - (-1.2816)(1.6449)}{1.6449 - (-1.2816)} \right) \approx 0.0232 \]

Accept sampled lot if for sample of \( n = 24 \), \( \bar{X} \geq L + \Delta_1 = 1.000 + 0.0232 = 1.0232 \) and reject the lot otherwise.

\[ p_1 = 0.03 = \Phi \left( \frac{1 - M_1}{0.015} \right) \Rightarrow M_1 = 1.0282 \]

\[ p_2 = 0.10 = \Phi \left( \frac{1 - M_2}{0.015} \right) \Rightarrow M_2 = 1.0192 \]

\[ Pa_1 = 1 - \Phi \left( \frac{1 + 0.0232 - 1.0282}{0.015/\sqrt{24}} \right) = 0.9488 \]

\[ Pa_2 = 1 - \Phi \left( \frac{1 + 0.0232 - 1.0192}{0.015/\sqrt{24}} \right) = 0.0512 \]

Now pick other \( M \)'s around/between \( M_1, M_2 \) & get \( p \) & \( Pa \) \( \rightarrow \) plot \( Pa \) vs. \( p \).

For attributes sampling plan use (8.19) V4J with \( \lambda_1 = p_2 = 0.03 \), \( \lambda_2 = p_2 = 0.10 \)

\[ \frac{\delta^2 (c_{n1})(1-\bar{p}_2)}{\delta^2 (c_{n1})(1-\lambda_1)} = \frac{\lambda_2}{\lambda_1} \]

Then find \( c_k \) (see page 457-458 V&J)

and then compare to variables sampling plan.
(b) find double limits sampling plan

\[ L = 1.49 \quad U = 1.51 \quad \sigma = 0.004 \quad P_{a1} = 0.03 \quad P_{a2} = 0.95 \]
\[ P_a = 0.10 \quad P_{a2} = 0.10 \]

\[ U - L = 1.51 - 1.49 = 0.02 \approx 6.0 = 0.24 \]
(i.e., \( U - L \) can be considered "large") or else follow \( P \) 471-472 V+J

\[ \Rightarrow \quad n = 25.85 \approx 24 \]
\[ \Delta z = 0.006187 \]

Accept sampled lot if for sample of \( n = 24 \), \( L + \Delta z \leq \bar{x} \leq U - \Delta z \)

\[ \Rightarrow 1.49 + 0.006187 \leq \bar{x} \leq 1.51 - 0.006187 \]

or reject the lot now.

\[ P_{a1} = 0.03 = 1 - \left[ \Phi \left( \frac{1.51 - \mu_1}{0.004} \right) - \Phi \left( \frac{1.49 - \mu_1}{0.004} \right) \right] \Rightarrow \text{can get } \mu_1 \quad \text{by interpolating Table 8.1}\]
\[ P_{a2} = 0.10 = 1 - \left[ \Phi \left( \frac{1.51 - \mu_2}{0.004} \right) - \Phi \left( \frac{1.49 - \mu_2}{0.004} \right) \right] \Rightarrow \text{can get } \mu_2 \]
\[ Pa = \Phi \left( \frac{1.51 - 0.006187 - \mu_1}{0.004} \right) - \Phi \left( \frac{1.49 + 0.006187 - \mu_1}{0.004} \right) = \# \quad \text{can get } \mu_1 \text{ above} \]
\[ Pa = \Phi \left( \frac{1.51 - 0.006187 - \mu_2}{0.004} \right) - \Phi \left( \frac{1.49 + 0.006187 - \mu_2}{0.004} \right) = \# \# \quad \text{can get } \mu_2 \]

Now pick some other \( \mu_1 \)s between \( \mu_1 \) and \( \mu_2 \) & get p & Pa \( \rightarrow \) plot Pa vs p for attributes sampling plan similarly use (8.19) V+J as before

\[ \frac{0.10}{0.03} = \frac{14}{3} \quad \text{and find } c \& k \text{ using (8.20), (8.21),}\]

Note that interpolation using Table 8.2 can be used.

[Check back by using formula (8.8)]
Then compare to Variables Sampling plan #
(c). Use Wallé approximation to find a single limit variable sampling plan for \( L = 1,000 \), \( \pi_1 = .03 \), \( \pi_2 = .10 \), \( \pi_1 = .95 \), \( \pi_2 = .10 \)

\[
K = \frac{\Phi z(\pi_1) - \Phi z(\pi_2)}{\Phi z(1-\pi_1) - \Phi z(1-\pi_2)}
\]

\[
= \frac{(1.6449)(1.2816) - (-1.2816)(1.8808)}{1.6449 - (-1.2816)} = 1.544
\]

\[
N \approx \left(1 + \frac{K^2}{6}\right) \left[\frac{\Phi z(\pi_1) - \Phi z(\pi_2)}{\Phi z(1-\pi_1) - \Phi z(1-\pi_2)}\right]^2
\]

\[
= \left(1 + \frac{1.544^2}{6}\right) \left(\frac{1.6449 - (-1.2816)}{1.8808 - (-1.2816)}\right)^2 \approx 52,287.0
\]

Accept sampled lot if for sample of size 52

\[
\bar{x} \geq L + K \bar{x} = 1 + 1.544\bar{x}
\]

\[
\begin{align*}
\pi_1 = .03 & = \Phi\left(\frac{L - \bar{x}}{\sigma}\right) \Rightarrow \frac{L - \bar{x}}{\sigma} = \Phi^{-1}(.03) = -1.88 \\
\pi_2 = .10 & = \Phi\left(\frac{L - \bar{x}}{\sigma}\right) \Rightarrow \frac{L - \bar{x}}{\sigma} = \Phi^{-1}(.10) = -1.2816
\end{align*}
\]

\[
\pi_a = 1 - \Phi\left(\frac{L - \bar{x} + K}{\sqrt{\frac{1}{N} + \frac{K^2}{2N}}}\right) = 1 - \Phi\left(\frac{-1.88 + 1.544}{\sqrt{\frac{1}{52} + \frac{1.544^2}{2(52)}}}\right) = 1 - \Phi(-1.6401) = .9475
\]

\[
\pi_a = 1 - \Phi\left(\frac{-1.2816 + 1.544}{\sqrt{\frac{1}{52} + \frac{(1.544)^2}{2(52)}}}\right) = .1004
\]

Now for other values of \( \frac{L - \bar{x}}{\sigma} \) around/between -1.88 and -1.2816 find \( \pi \) and \( \pi_a \). Then plot \( \pi_a \) vs. \( \pi \) to get OC curve.
5.6 Find \( n \) for known \( \sigma \) single limit variable acceptance sampling plan to have

\[
P_a = 0.95 \quad \text{if} \quad p = 10^{-6},
\]

\[
P_a = 0.10 \quad \text{if} \quad p = 3 \times 10^{-6}.
\]

\[
n \approx \left( \frac{\delta_e (p_a) - \delta_e (p_a - \delta_e (1 - p_a))}{\delta_e (1 - p_a) - \delta_e (1 - 3 \times 10^{-6})} \right)^2
\]

\[
= \left( \frac{1.6449 - (-1.2816)}{4.733 - 4.526} \right)^2 = 166.2055
\]

We are discriminating by a very small fraction nonconforming. We are relying heavily on appropriateness of normal model in the tails of the real dist. of measurements.

5.7 Variable acceptance sampling based on exp. dist'd obsr.

Single lower limit \( l = 0.2107 \)

(a) \( X \sim \text{exp}(\lambda) \), \( \lambda > 0 \) with mean \( \lambda \)

\[
\lambda x(x) = \begin{cases} 
\frac{1}{\lambda} e^{-x/\lambda} & \text{for } x > 0 \\
0 & \text{otherwise}
\end{cases}
\]

\[
P = P(X < 0.2107)
\]

\[
= \int_0^{0.2107} \frac{1}{\lambda} e^{-x/\lambda} \, dx = \left[ -e^{-x/\lambda} \right]_0^{0.2107} = 1 - e^{-0.2107/\lambda}
\]

\[
\Rightarrow \lambda(p) = \frac{-0.2107}{\ln(1-p)}
\]

\[
\lambda(0.10) = 1.9998 \approx 2
\]

\[
\lambda(0.01) = 0.9999 \approx 1
\]
(b). Find \( n, k \) for acceptance sampling plan that rejects a lot if \( \bar{x} < k \) has \( \alpha = .05 \) for \( \beta = .10 \) and \( \alpha = .10 \) for \( \beta = .019 \).

For large \( n \), \( \bar{x} \sim N(\lambda, \frac{\lambda^2}{n}) \)

\[
P(\bar{x} > k) = .05
\]

\[
P\left( z > \frac{k - \lambda}{\lambda / \sqrt{n}} \right) = .05
\]

\[
\Rightarrow \frac{k - \lambda}{\lambda / \sqrt{n}} = -1.645
\]

\( \lambda (.05) = 2 \)

\[
\Rightarrow \frac{2 - \lambda}{\lambda / \sqrt{n}} = -1.645 \quad (*)
\]

Also, \( P(\bar{x} > k) = .10 \)

\[
P\left( z > \frac{k - \lambda}{\lambda / \sqrt{n}} \right) = .2816
\]

\( \lambda (.10) = 1 \)

\[
\Rightarrow \frac{k - 1}{\lambda / \sqrt{n}} = 1.2816 \quad (**)
\]

From (*) and (**)

\[
\frac{k - 2}{-1.645} = 2 \left( \frac{k - 1}{1.2816} \right)
\]

\[
1.2816 (k - 2) = 2 (-1.645) (k - 1)
\]

\[
\Rightarrow k = 1.2803
\]

\[
\Rightarrow \frac{1.2803 - 2}{\sqrt{n}} = -1.645 \Rightarrow \text{solve for } n
\]

\[
\Rightarrow \frac{1.2803 - 1}{\sqrt{n}} = 1.2816 \Rightarrow \text{solve for } n
\]
(c) \[ P_a = P(\bar{X} > 1.28) \]

\[ \approx 1 - \Phi \left( \frac{1.28 - \frac{2.107}{\sqrt{n} \ln(1-p)}}{\sqrt{\frac{2.107}{\ln(1-p)}}} \right) \]

\[ = 1 - \Phi \left( \frac{\sqrt{5} \ln(1-p) \left[ 1.28 + \frac{2.107}{\ln(1-p)} \right]}{-2.107} \right) \]

\[ = 1 - \Phi \left( \frac{1.28 \sqrt{5} \ln(1-p) + 2.107 \sqrt{5}}{-2.107} \right) \]

\[ = 1 - \Phi \left( -13.584 \ln(1-p) - 2.2361 \right) \]

<table>
<thead>
<tr>
<th>( p )</th>
<th>( P_a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.9873</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9133</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1873 \times 10^{-6}</td>
</tr>
<tr>
<td>0.1</td>
<td>0.7896</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0045</td>
</tr>
<tr>
<td>0.15</td>
<td>0.5115</td>
</tr>
</tbody>
</table>

\[ \Rightarrow \text{plot it!} \]
(a) An appropriate expected total cost is

\[ \text{ETC} = k_1 n + (N-n)(1-p_a)k_1 + (N-n) p_a \left[ p_m k_2 + p_p k_3 \right] \]

\[ = N k_1 + (N-n) p_a \left[ p_m k_2 + p_p k_3 - k_1 \right] \]

It is only the 2nd term that we may influence by choice of \( n \) (and the decision criterion). When

\[ p_m k_2 + p_p k_3 > k_1 \]

this term is positive and we wish to remove it from the ETC. This can be done by choice of \( n = N \) i.e. "All". On the other hand, when

\[ p_m k_2 + p_p k_3 < k_1 \]

the term is negative and we wish to make it's multiplier as large as possible. This can be done by choice of \( n = 0 \) and \( p_a = 1 \), i.e. "None".

So in case \( \ast \) "All" is optimal, while in case \( \ast \ast \) "None" is optimal.

(b) With \( n \) inspections made, and counts \( X_0, X_m \) and \( X_D \) in hand, with rejection one has conditional expected total cost

\[ N k_1 \]

while with acceptance, the conditional expected total cost is

\[ \text{ETC} = k_1 E \left[ k_2 E \left[ (N-n) \left( \frac{k_2}{x_0 + x_m + x_D} + k_2 \frac{x_D + x_p}{x_0 + x_m + x_D} \right) \right] \right] \]

and one should reject the lot if \( \ast \) is less than \( \ast \ast \) i.e. if

\[ \text{if} \]

\[ k_1 < k_2 \frac{x_m + x_D}{x_0 + x_m + x_D + n} + k_3 \frac{x_D + x_p}{x_0 + x_m + x_D + n} \]
We know that \( C_{6}^{\text{opt}}(n) \) for given \( n, \alpha, \beta, k_1, k_2 \) is the largest \( x \) \( \exists \)

\[
\frac{\alpha + x}{\alpha + \beta + n} \leq \frac{k_1}{k_2}
\]

i.e. \( C_{6}^{\text{opt}}(n) = \left\lfloor \frac{k_1}{k_2} n - \alpha + \frac{k_1}{k_2} (\alpha + \beta) \right\rfloor \)

Here \( E_6(1) = 0.1 = \frac{\alpha}{\alpha + \beta} \)

\[\text{Var}_6(\rho) = \frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)} = (0.090533)^2\]

\[
\Rightarrow \frac{\rho}{\alpha + \beta} = 0.1 \Rightarrow \frac{\alpha \beta}{(\alpha + \beta)^2} = 0.09 \Rightarrow \alpha + \beta + 1 = 11 \Rightarrow \alpha + \beta = 10
\]

\[
\Rightarrow \alpha = 1, \beta = 9
\]

\[
\therefore \bigcirc \Rightarrow C_{6}^{\text{opt}}(n) = \left\lfloor \frac{n}{10} - 1 + \frac{1}{9} n \times 10 \right\rfloor = \left\lfloor \frac{n}{10} \right\rfloor
\]

\( c = 0 \) is optimal \( \iff \)

\( c = 1 \quad 1 \leq n < 10 \)
\( c = 2 \quad 10 \leq n < 29 \)

Expected cost for \( (n, C_{6}^{\text{opt}}(n)) \)

\[
= k_1 N + k_1 N (1 - \frac{D}{N}) \sum_{x=0}^{C_{6}^{\text{opt}}(n)} (\frac{n}{x}) \int_0^1 (p \frac{k_2}{k_1} - 1) p (1-p)^{n-x-1} \frac{d}{B(\alpha, \beta) B(1-\beta)} dp
\]

It can be found that \( n = 25, c = 2 \) is optimal

\( \text{Exp Cost} (n = 25, c = 2, \$7407.59) \)
\[ N = 100, \quad k_1 = 1, \quad k_2 = 10 \]
\[
E \text{ Total Cost (n,c,p)} = k_1 N \left( 1 + P_a(n,c,p) \left( 1 - \frac{n}{N} \right) \left( \frac{k_1}{k_1} \right) - 1 \right)
\]
\[
= 100 \left( 1 + P_a(n,c,p) \left( 1 - \frac{n}{100} \right) (10p - 1) \right)
\]
\[
\text{where} \quad P_a = \sum_{x=0}^{c} \left( \begin{array}{c} n \\ x \end{array} \right) p^x (1-p)^{n-x}
\]

Let \( \Delta(p) \equiv \frac{\text{worst possible } E \text{ total cost (p)} - \text{best possible } E \text{ total cost (p)}}{\text{best possible } E \text{ total cost (p)}} \)

\[
100 \left[ 1 + \left( \sum_{x=0}^{c} \left( \begin{array}{c} n \\ x \end{array} \right) p^x (1-p)^{n-x} \right) \left( \frac{n}{N} \right) (10p - 1) \right] - 100 \left[ 1 + \left( \sum_{x=0}^{c} \left( \begin{array}{c} n \\ x \end{array} \right) p^x (1-p)^{n-x} \right) \left( \frac{n}{N} \right) (10p - 1) \right]
\]

\[
\Delta(p) = \frac{100 \left[ 1 + \left( \sum_{x=0}^{c} \left( \begin{array}{c} n \\ x \end{array} \right) p^x (1-p)^{n-x} \right) \left( \frac{n}{N} \right) (10p - 1) \right]}{100 \left[ 1 + \left( \sum_{x=0}^{c} \left( \begin{array}{c} n \\ x \end{array} \right) p^x (1-p)^{n-x} \right) \left( \frac{n}{N} \right) (10p - 1) \right]}
\]

\[ p < 0.1 \Rightarrow 10p - 1 < 0 \Rightarrow n = 0 \quad \text{or} \quad "\text{None}" \quad \text{is optimal} \]
\[ p > 0.1 \Rightarrow 10p - 1 > 0 \Rightarrow n = N \quad \text{or} \quad "\text{All}" \quad \text{is optimal} \]

\[ p < 0.1: \quad \text{Best possible plan has } n = 0 \quad \text{and} \quad \Delta(p) = 1 \]
\[ \text{Worst} \quad 10p \quad \text{is minimal} \]
\[ \Delta(p) = \frac{k_1 N - 10p k_1 N}{10p k_1 N} = 1 - \frac{10p - 1}{10p} \]

As \( p \) ranges from \([0.0, 0.1]\), this \( \Delta(p) \) reduces from 0.99 to 0.

\[ p = 0.1: \quad \Delta(p) = 0 \]

\[ p > 0.1: \quad \text{Best possible plan has } n = N \quad \text{and} \quad \Delta(p) = \frac{10p k_1 N - k_1 N}{k_1 N} = 10p - 1 \]

\[ \text{Worst} \quad 10p \quad \text{is minimal} \]

\[ \Delta(p) = \frac{10p k_1 N - k_1 N}{k_1 N} = 10p - 1 \]
As \( p \) takes values in \((0.1, 0.11]\), this ratio increases from \((0, 0.1)\).

So, this quantity (or ratio) never gets larger than 0.11 in the given interval, so there is not much difference between the worst possible ETC and the best possible ETC.