Inference? / Confidence Limits?

\[ C_{pk} = \min \left\{ \frac{U - \mu}{3\sigma}, \frac{\mu - L}{3\sigma} \right\} \]

the "number of 's-sigmas' that the mean is to the good side of the nearest specification".

\[ L \quad \frac{\bar{X} - L}{3\sigma} \quad U \]

\[ \bar{x}_1, \bar{x}_2, ..., \bar{x}_n \text{ iid } N(\mu, \sigma^2) \]

\[ \bar{X}, \sigma \]

\[ \hat{C}_{pk} = \frac{\bar{X} - L}{3\sigma} \]

\[ \hat{C}_{pk} = \frac{U - \bar{X}}{3\sigma} \]

\[ \hat{C}_{pk} - z\sqrt{\frac{1}{n} + \frac{\hat{C}_{pk}^2}{2(n-1)}} \]

\[ \Delta \text{ method standard error} \]

\[ \text{Var } g(\bar{X}, \sigma^2) \text{ can be approximated using } \Delta \text{ method} \]

Suppose \( \mu > \frac{L + U}{2} \)

\[ \text{Var } \hat{C}_{pk} \approx \left( \frac{1}{3\sigma} \right)^2 \text{Var } \bar{X} \]

\[ + \left( \frac{U - \mu}{3\sigma} \right) \left( -\frac{1}{2} \left( \sigma^2 \right)^{-2} \right) \text{Var } \sigma^2 \]

now plug in \( \text{Var } \bar{X} = \frac{\sigma^2}{n} \)

\[ \text{Var } \sigma^2 = \frac{\sigma^4}{n-1} \]

and simplify to get

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\[
\frac{1}{S^2_n} + \frac{C_{pk}}{2(n-1)}
\]

plug in \( \hat{C}_{pk} \) for \( C_{pk} \) and take the root and you have a standard error for \( \hat{C}_{pk} \).

3.3 Another way to characterize process output is through statements of where individual values are likely to be - prediction and tolerance intervals.

**PI problem:**

If \( P_F[ X_{n+1} \in I(\hat{X})] \geq \delta \quad \forall F \in \mathcal{F} \)

Then I call \( I(\hat{X}) \) a \( \delta \)-level PI

**TI problem:**

If \( P_F[ \max(F(g_u(X)) - F(g_L(X)) \geq \delta] \geq \delta \)

\( \forall F \in \mathcal{F} \) then I call \( I(\hat{X}) \) a \( \delta \)-level TI for a fraction \( p \) of the underlying distribution \( F \)

**Simpler Version:**

This is where \( F = \text{class of all cont. dsns} \).

\[ X_1, X_2, \ldots, X_n \text{ iid } F \in \mathcal{F} \]

some class of cont. dsns

make up a random interval \( (g_L(X), g_U(X)) = I(\hat{X}) \)

from \( \hat{X} = (X_1, \ldots, X_n) \)

\[ X_{n+1} \]

\[ \begin{array}{c|c|c|c}
\text{Interval} & \text{Confidence Level} & \text{TI for fraction} \\
\hline
(\min X_i, \infty) & \frac{n}{n+1} & 1 - p^n \\
(-\infty, \max X_i) & \frac{n}{n+1} & 1 - p^n \\
(\min X_i, \max X_i) & \frac{n-1}{n+1} & 1 - p^n - n(1-p)p^{n-1} \\
\end{array} \]
One-Sided Intervals

As PI's:

\[ P \left[ \min \left( X_1, \infty \right) \text{ doesn't cover } X_{n+1} \right] \]
\[ = P \left[ X_{n+1} \text{ is smallest of } X_1, X_2, \ldots, X_{n+1} \right] \]
\[ = \frac{n}{n+1} \]

\[ \text{confidence level is } \frac{n}{n+1} \text{ for prediction} \]

As TI's for a fraction \( p \) of a dataset/process/population

Two-Sided Interval

As PI:

\[ P \left[ \left( \min X_i, \max X_i \right) \text{ doesn't cover } X_{n+1} \right] \]
\[ = P \left[ X_{n+1} \text{ is the smallest or largest of } X_1, X_2, \ldots, X_{n+1} \right] \]
\[ = \frac{2}{n+1} \text{ by symmetry} \]

\[ \text{confidence level is } \frac{n-1}{n+1} \]

As TI:

\[ P \left[ F(x) \leq t \right] = P \left[ X \leq F^{-1}(t) \right] = F \left( F^{-1}(t) \right) = t \]

\[ \text{i.e. } F(x) \text{ is } U(0,1) \]
For \( r \in (0, 1) \)

\[
f(r) = \int_0^{1-r} n(n-1) r^{n-2} \, dr
\]

Hence

\[
P[R \geq p] = \int_0^{1-p} f(r) \, dr
\]

\[
= 1 - p^n - n(1-p)p^{n-1}
\]

There is also a normal distribution technology for PI's + TI's

**PI stuff**

\[
X_1, \ldots, X_n, X_{n+1} \text{ iid } N(\mu, \sigma^2)
\]

\[
\bar{X} \sim N(0, \sigma^2(1 + \frac{1}{n}))
\]

\[
\frac{(n-1)\hat{S}^2}{\sigma^2} \sim \chi^2_{n-1}
\]

Independent

\[
T = \frac{X_{n+1} - \bar{X}}{S \sqrt{1 + \frac{1}{n}}}
\]

So if \( t^* \) and \( t^{**} \) are such that

\[
P\left[ t^* < \frac{\bar{X} - \mu}{\sigma \sqrt{\frac{1}{n}}} < t^{**} \right] = \gamma
\]
\[ P \left[ \frac{t^*}{\sqrt{1 + \frac{1}{n}}} < T < \frac{t**}{\sqrt{1 + \frac{1}{n}}} \right] = \gamma \]

\[ X + t^* s / \sqrt{1 + \frac{1}{n}} < \bar{X} < X + t** s / \sqrt{1 + \frac{1}{n}} \]

i.e.

\[ \bar{X} \pm t s / \sqrt{1 + \frac{1}{n}} \]

are normal theory prediction limits for \( \bar{X}_{n+1} \)

\[ X + t s \]

\[ \text{quantile} \]

The dsn

\[ P[\bar{X} + t s \geq \text{quantile}] \geq \gamma \]

\[ X - \mu - z_p \sigma \geq -t s \]

\[ \frac{X - \mu}{s / \sqrt{1 + \frac{1}{n}}} - \sqrt{n} z_p \geq -t \]

\[ \frac{1}{\sqrt{n}} \left( \frac{X - \mu}{s / \sqrt{1 + \frac{1}{n}}} - \sqrt{n} z_p \right) \geq -t \]

\[ a N(q(1)), a / \chi^2_{n-1} \text{ r.v.} \]

\[ a \chi^2_{n-1} \text{ r.v.} \]

\[ \text{independent} \]

\[ a \text{ noncentral } \chi^2_{n-1} \text{ r.v. with noncentrality parameter } -\sqrt{n} z_p \]

i.e. I want

\[-\sqrt{n} t = 1 - \gamma \text{ quantile of the noncentral } \chi^2_{n-1}(\sqrt{n} z_p)\]
2) \( \exists \) two-sided one-sample T I's

\[
\left( \bar{X} - \mu, \bar{X} + \mu \right)
\]

\[
P_{\alpha} \left[ \Phi \left( \frac{\bar{X} + \mu - \mu}{\sigma} \right) - \Phi \left( \frac{\bar{X} - \mu - \mu}{\sigma} \right) \geq \alpha \right] \geq \gamma
\]

Notes:
1) \( \exists \) explicit approximations for T's just considered - See Problem 4.3 of Notes

This is the same as

\[
P \left[ \Phi \left( \frac{\bar{X} + \mu + \sqrt{\frac{W}{n-1}}} {\sqrt{n}} \right) - \Phi \left( \frac{\bar{X} + \mu - \sqrt{\frac{W}{n-1}}} {\sqrt{n}} \right) \geq \alpha \right] \geq \gamma
\]

where \( \bar{X} \sim \text{std normal} \) independent

\( W \sim \chi^2_{n-1} \)

I must tweak \( T \) to get the desired \( \gamma \)

\( \Phi \) - See Table A.3.A of V+J

5.4 Probabilistic Tolerance/Error Analysis

Technically this is nothing more than an application of propagation of error (or simulation) - application is to engineering design - idea is that in

said contexts there are good deterministic models for how an output is a function of several inputs

\[ U = g \left( X, Y, ..., Z \right) \]

Some product - Properties of the product

quality variable
Example 5.9

• Nice little tolerance stack-up problem

\[ U = Y - X_1 - X_2 - X_3 - X_4 \]

\[ \sigma_U^2 = 1^2 \sigma_Y^2 + (-1)^2 \sigma_{X_1}^2 + (-1)^2 \sigma_{X_2}^2 + (-1)^2 \sigma_{X_3}^2 + (-1)^2 \sigma_{X_4}^2 \]

• \( U \) was head space in a carton designed to hold 4 units of product
Example 5.8

\[ R = R_1 + \frac{R_2 R_3}{R_2 + R_3} \]

\[ \mu_{R_1} = 100\Omega \text{ and } \mu_{R_2} = \mu_{R_3} = 200\Omega \]

\[ \sigma_{R_1} = 2\Omega \text{ and } \sigma_{R_2} = \sigma_{R_3} = 4\Omega \]

What about \( R \)?
Given nominals and “sigmas” for \( x, y, w, \phi, \theta_1 \) and \( \theta_2 \) what can I say about the gaps, \( g_1 \) and \( g_2 \)?

\[ p = (-x \sin \phi, x \cos \phi) \]
\[ q = p + \left( y \cos \left( \phi + \left( \theta_1 - \frac{\pi}{2} \right) \right), y \sin \left( \phi + \left( \theta_1 - \frac{\pi}{2} \right) \right) \right) \]
\[ s = (q_1 + q_2 \tan(\phi + \theta_1 + \theta_2 - \pi), 0) \]
\[ u = (q_1 + (q_2 + d) \tan(\phi + \theta_1 + \theta_2 - \pi), -d) \]
\[ g_1 = w - s_1 \]
\[ g_2 = w - u_1 \]
Taylor series analysis says for independent inputs
\[
\text{Var } U \approx \left( \frac{\partial^2 g}{\partial x^2} \right)^2 \text{Var } X + \left( \frac{\partial^2 g}{\partial y^2} \right)^2 \text{Var } Y \\
+ \ldots + \left( \frac{\partial^2 g}{\partial z^2} \right)^2 \text{Var } Z
\]

(where partials are evaluated at \((EX, EY, \ldots, EZ)\)). I can therefore hope to make \(\text{Var } U\) small by controlling \(\text{Var } X, \text{Var } Y, \ldots, \text{Var } Z\) or by making partials small (possibly by choice of \(EX, EY, \ldots, EY\)).

Section 5.5 of V+J (Ch 4 of notes)

Variance Component Analysis in Hierarchical Contexts:

A common structure in industrial contexts is one where levels of A give rise to levels of B, which give rise to levels of C, etc. and one would like to "partition" variability encountered.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Months</td>
<td>Weeks</td>
<td>Days</td>
<td>Shifts</td>
</tr>
<tr>
<td>Heats</td>
<td>Ingots</td>
<td>Specimens</td>
<td>Lab Determinations</td>
</tr>
</tbody>
</table>

Robust ("Taguchi") product design

see problems 4.11 and 4.12 of notes for "Examples"
\[ y_{ijk} = \mu + x_i + \beta_{ij} + \epsilon_{ijk} \]

for \( x_i, \beta_{ij} \) and \( \epsilon_{ijk} \) independent

\[ x_i \sim N(0, \sigma_x^2) \]
\[ \beta_{ij} \sim N(0, \sigma_\beta^2) \]
\[ \epsilon_{ijk} \sim N(0, \sigma^2) \]

For balanced data, estimation of variance components is standard and based on ANOVA. SS's and MS's - (3)
we've dealt with MS's before...