Predator-Prey Models
(Notes for Math 182)

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1 A second-order differential equation

Differential equations that involve the second derivative of the unknown function are called second-order differential equations. In this section we will learn to solve one family of such equations. The solutions will be useful for studying simple predator-prey models.

Theorem 1.1. Let $\omega$ be a positive constant. The differential equation

$$y''(t) + \omega^2 y(t) = 0$$

has for its most general solution the function

$$y(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t)$$

where $C_1$ and $C_2$ are arbitrary constants.

Proof. Suppose

$$y(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t).$$

Then we compute

$$y'(t) = C_1 \omega \cos(\omega t) - C_2 \omega \sin(\omega t).$$

Differentiate this expression again:

$$y''(t) = -C_1 \omega^2 \sin(\omega t) - C_2 \omega^2 \cos(\omega t).$$

But by multiplying $y(t)$ by $\omega^2$ we obtain

$$\omega^2 y(t) = C_1 \omega^2 \sin(\omega t) + C_2 \omega^2 \cos(\omega t).$$
Adding the two equations we find
\[ y''(t) + \omega^2 y(t) = -C_1 \omega^2 \sin(\omega t) - C_2 \omega^2 \cos(\omega t) + C_1 \omega^2 \sin(\omega t) + C_2 \omega^2 \cos(\omega t) \]
Hence
\[ y''(t) + \omega^2 y(t) = 0. \]
This argument shows that the formula in the theorem does in fact solve the differential equation, no matter what the values of the constants \( C_1 \) and \( C_2 \) may be. It turns out that the formula does in fact give all the solutions of the differential equation, but that part of the proof is omitted here.

Example. Tell the most general solution to \( y''(t) + 9y(t) = 0 \).

Solution. Utilize the theorem. The differential equation given in the problem matches that of the theorem when \( \omega = 3 \). Hence the theorem says that the most general solution is \( y(t) = C_1 \sin(3t) + C_2 \cos(3t) \).

Example. Find the solution of \( y''(t) + 4y(t) = 0 \) that satisfies \( y(0) = 7 \), \( y'(0) = 3 \).

Solution. The differential equation in the problem matches that of the theorem when \( \omega = 2 \). Hence every solution \( y(t) \) has the form
\[ y(t) = C_1 \sin(2t) + C_2 \cos(2t). \]
But \( y(0) = 7 \), while the formula says
\[ y(0) = C_1 \sin(2(0)) + C_2 \cos(2(0)) = C_1(0) + C_2(1) = C_2. \]
Hence \( C_2 = 7 \). Differentiate the formula for \( y(t) \):
\[ y'(t) = C_1(2) \cos(2t) - C_2(2) \sin(2t). \]
But \( y'(0) = 3 \), while the formula says
\[ y'(0) = C_1(2) \cos(2(0)) - C_2(2) \sin(2(0)) = C_1(2)(1) - C_2(2)(0) = 2C_1 \]
Hence \( 2C_1 = 3 \), so \( C_1 = 3/2 \). The answer is
\[ y(t) = (3/2) \sin(2t) + 7 \cos(2t). \]

Example. Find the solution of
\[ u''(t) + 3u(t) = 0 \]
that satisfies \( u(0) = 1 \), \( u'(0) = 2 \).

Solution. The differential equation in the problem matches that of the theorem when \( u \) corresponds to \( y \) and \( \omega = \sqrt{3} \). Hence every solution \( u(t) \) has the form
\[ u(t) = C_1 \sin(\sqrt{3}t) + C_2 \cos(\sqrt{3}t). \]
But $u(0) = 1$, while the formula says $u(0) = C_1(0) + C_2(1) = C_2$. Hence $C_2 = 1$.
The derivative of the formula for $u(t)$ is

$$u'(t) = C_1(\sqrt{3}) \cos(\sqrt{3}t) - C_2(\sqrt{3}) \sin(\sqrt{3}t).$$

But $u'(0) = 2$, while the formula says

$$u'(0) = C_1(\sqrt{3})(1) - C_2(\sqrt{3})(0) = \sqrt{3}C_1.$$

Hence $\sqrt{3}C_1 = 2$, so $C_1 = 2/\sqrt{3}$.
The answer is $u(t) = (2/\sqrt{3}) \sin(\sqrt{3}t) + \cos(\sqrt{3}t)$.

2 Simple predator-prey models

The exponential and logistic models of population growth both dealt with one species at a time and ignored any interactions with other species. For example, the logistic model for a fish population ignored any specific predation by other species (such as larger fish, whales, or seals) that might eat the fish.

Suppose we regard a population of fish as the "prey" and a population of seals as the "predators." We assume that the fish are the only food eaten by the seals. If the seal population grows substantially, there will be an immediate effect on the number of fish because the larger population of seals will eat substantially more fish. So the fish population will fall. On the other hand, when the fish population drops, then this will in turn affect the number of seals, since more seals will starve to death as a result of the smaller number of fish. The smaller number of seals could then allow more fish to survive since the smaller number of seals would eat fewer fish. But then the fish population will rise again, so the abundance of food will permit more seals to survive and the seal population could grow back.

This discussion shows that there is a possibility of cyclical rise and fall in the populations of the seals and the fish. The sizes of both the seal population and the fish population are interrelated. This section describes a simple way in which such an interrelationship can sometimes be described mathematically.

We consider a system consisting of a population of prey (like fish) and a population of predators (like seals). Let $Y(t)$ be the number of prey at the time $t$, and let $D(t)$ be the number of predators at the time $t$. Over the long term there will be an average size of the prey population and also an average size of the predator population; but the actual populations will probably oscillate around these average values. Write $Y_e$ for the average long-term size of the prey population, and write $D_e$ for the average long-term size of the predator population. These numbers will be regarded as constant.

Let $y(t) = Y(t) - Y_e$ and $d(t) = D(t) - D_e$. These quantities tell the deviation of the populations at time $t$ from their average value. We call $y(t)$ the excess prey population, and we call $d(t)$ the excess predator population. For example, if $y(3) > 0$, then $Y(3) - Y_e > 0$, so $Y(3) > Y_e$, so at time 3 the prey population is larger than usual. If $y(5) < 0$ then $Y(5) - Y_e < 0$, so $Y(5) < Y_e$. 

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and at time \( t = 5 \) the prey population is smaller than usual. Similarly if \( d(5) > 0 \) then \( D(5) - D_\epsilon > 0 \), so \( D(5) > D_\epsilon \) and at time 5 the predator population is larger than usual.

The model that we will utilize is as follows: Assume there are positive constants \( a \) and \( b \) such that

\[
y'(t) = -ad(t)
\]
\[
d'(t) = by(t).
\]

Call \( a \) the \textbf{growth parameter for the prey} (since it tells about \( y'(t) \), the rate of growth related to the prey population), and call \( b \) the \textbf{growth parameter for the predators} (since it tells about \( d'(t) \), which is the rate of growth related to the predator population).

This says that if \( y(t) > 0 \) (so that there are more prey than usual), then the population of predators will increase (\( d'(t) = by(t) \) will be positive, and a positive derivative means that the numbers will increase). On the other hand, if \( y(t) < 0 \) (so there are fewer prey than usual), then the population of predators will decrease. (This is because \( d'(t) = by(t) \), so if \( y(t) < 0 \) then \( d'(t) < 0 \) and the number of predators will decrease.)

Similarly if \( d(t) > 0 \) (so that there are more predators than usual), then \( y'(t) = -ad(t) \) so \( y'(t) < 0 \) and \( y(t) \) will be a decreasing function; this means that when there are more predators than usual, then the number of prey will decrease. On the other hand, if \( d(t) < 0 \) (so that there are fewer predators than usual), then \( y'(t) = -ad(t) \) says that \( y'(t) > 0 \) so \( y(t) \) will increase.

Thus the two short equations describe very concisely our understanding of how the abundance of prey affects the numbers of predators, and also how the abundance of predators affects the number of prey.

\textbf{Theorem 2.1.} Let \( \omega = \sqrt{ab} \). There are constants \( C_1, C_2, C_3, \) and \( C_4 \) such that

\[
Y(t) = Y_\epsilon + C_1 \sin(\omega t) + C_2 \cos(\omega t)
\]
\[
D(t) = D_\epsilon + C_3 \sin(\omega t) + C_4 \cos(\omega t)
\]

This result shows how the sine and cosine functions are needed to describe population sizes, at least in this case. It is initially a surprise—after all, what do right triangles and angles have to do with population sizes? Ultimately, such functions are useful to biologists because they describe oscillations in systems. There are lots of biological systems that show oscillation over time, and the predator-prey systems discussed here are one example.

How do these formulas arise? Rewrite the first equation:

\[
y'(t) = -ad(t)
\]

Differentiate both sides; since we are doing the same thing to equals, we get a new equality:

\[
y''(t) = -ad'(t)
\]
But from the second equation we know \( d'(t) = by(t) \). Substituting, we see

\[ y''(t) = -a(by(t)) \]

Hence \( y''(t) + aby(t) = 0 \). We already know how to solve this differential equation. There must be constants \( C_1 \) and \( C_2 \) such that when \( \omega = \sqrt{ab} \) then

\[ y(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t) \]

But \( y(t) = Y(t) - Y_e \), so

\[ Y(t) - Y_e = C_1 \sin(\omega t) + C_2 \cos(\omega t) \]

\[ Y(t) = Y_e + C_1 \sin(\omega t) + C_2 \cos(\omega t). \]

A similar argument yields the result for \( D(t) \).

Here is an example that illustrates the mathematical methods involved in solving these predator-prey models.

**Example.** In a certain region, the excess population of rabbits is \( r(t) \), while the excess population of wolves is \( w(t) \). Suppose \( r'(t) = -3w(t) \), \( w'(t) = 12r(t) \), \( r(0) = 3000 \), and \( w(0) = 500 \). Find \( r(t) \) and \( w(t) \).

**Solution.** \( r'(t) = -3w(t) \). Hence

\[ r''(t) = -3w'(t) = -3(12r(t)) = -36r(t). \]

Now \( r''(t) + 36r(t) = 0 \), so that

\[ r(t) = C_1 \sin(6t) + C_2 \cos(6t). \]

Since \( r(0) = 3000 \), we have \( C_1(0) + C_2(1) = 3000 \) so \( C_2 = 3000 \).

Hence \( r(t) = C_1 \sin(6t) + 3000 \cos(6t) \)

\[ \infty r'(t) = 6C_1 \cos(6t) - 18000 \sin(6t) \]

But \( r'(t) = -3w(t) \) so \( r'(0) = -3w(0) = -3(500) = -1500 \).

Hence \( r'(0) = 6C_1 \cos(6(0)) - 18000 \sin(6(0)) = -1500 \)

\[ 6C_1 = -1500 \]

\[ C_1 = -250 \]

It follows that \( r(t) = -250 \sin(6t) + 3000 \cos(6t) \) rabbits.

But since \( r'(t) = -3w(t) \), we see that \( w(t) = r'(t)/(-3) \). Now

\[ r'(t) = -1500 \cos(6t) - 18000 \sin(6t), \]

so

\[ w(t) = r'(t)/(-3) = 500 \cos(6t) + 6000 \sin(6t) \] wolves.

The next example shows an entire problem.

**Example.** In a certain area in the Southwestern United States there is a population of squirrels and coyotes. Regard the squirrels as prey and the coyotes as predators since the coyotes eat squirrels but the squirrels do not eat coyotes. On average the population contains 300 thousand squirrels and 50 thousand coyotes, but in fact the actual populations oscillate around these numbers. Time is measured in months. Measurements over time show that the growth parameter for the prey is \( a = 8 \), while the growth parameter for the predators is \( b = 2 \). A careful survey at the base time (which will be called \( t = 0 \)) showed that the population of squirrels was 310 thousand while the population of coyotes was 45 thousand. Tell formulas for the populations of both squirrels and coyotes for all time \( t \).
Solution. Let \( S(t) \) be the number of thousands of squirrels in the population during the \( t \)-th month after the base time, while \( C(t) \) is the number of thousands of coyotes during the \( t \)-th month after the base time. The average value of \( S \) is \( S_e = 300 \), while the average value of \( C \) is \( C_e = 50 \). Hence the excess squirrel population is \( s(t) = S(t) - S_e = S(t) - 300 \), and the excess coyote population is \( c(t) = C(t) - C_e = C(t) - 50 \).

Since the growth parameter for the prey is \( a = 8 \) we know \( s'(t) = -8e(t) \).
Since the growth parameter for the predators is \( b = 2 \), we know \( c'(t) = 2s(t) \).
Moreover, \( s(0) = S(0) - 300 = 310 - 300 = 10 \), while \( c(0) = C(0) - 50 = 45 - 50 = -5 \). With this information, we can solve for \( s(t) \) and \( c(t) \).

To do this we note \( s'(t) = -8e(t) \), so \( s''(t) = -8e'(t) = -8(2s(t)) = -16s(t) \).

Hence
\[
s''(t) + 16s(t) = 0
\]
\[
s(t) = K_1 \sin(4t) + K_2 \cos(4t).
\]

But \( s(0) = K_1 \sin(0) + K_2 \cos(0) = 10 \), so \( K_2 = 10 \).
Since \( s'(t) = -8e(t) \), it follows that \( s'(0) = -8(-5) = 40 \). But
\[
s'(0) = 4K_1 \cos(0) - 4K_2 \sin(0) = 40 \)
\[
K_1 = 40 \quad \text{and} \quad K_1 = 10. \]

Hence
\[
s(t) = 10 \sin(4t) + 10 \cos(4t).
\]

Now \( c(t) = s'(t)/(-8) = (40 \cos(4t) - 40 \sin(4t))/(-8) = -5 \cos(4t) + 5 \sin(4t) \).

To find \( S(t) \) we recall \( s(t) = S(t) - 300 \), so
\[
S(t) = 300 + s(t) = 300 + 10 \sin(4t) + 10 \cos(4t).
\]

Similarly, since \( c(t) = C(t) - 50 \), it follows that
\[
C(t) = 50 + c(t) = 50 + 5 \sin(4t) - 5 \cos(4t).
\]

According to this model, in the \( t \)-th month there are \( 300 + 10 \sin(4t) + 10 \cos(4t) \) thousand squirrels as well as \( 50 + 5 \sin(4t) - 5 \cos(4t) \) thousand coyotes.

The graphs of \( S(t) \) and \( C(t) \) are shown in Figure 1, where \( S(t) \) is the upper curve and \( C(t) \) is the lower curve. Note that both of the curves show oscillation. These oscillations arise because both the sine and the cosine functions exhibit oscillation. A careful look shows that the two curves do not, however, reach their peaks at the same moment. When the prey is maximal (at a peak of the upper curve), the predators are increasing in number (the lower curve is still moving up).

Example. Continuing with the squirrel and coyote example, tell the maximum and minimum number of squirrels. Solution. We want to maximize or minimize \( S(t) \). To find the maximum or minimum we solve the equation
\[
S''(t) = 0. \]

Then
\[
40 \cos(4t) - 40 \sin(4t) = 0 \]
\[
\cos(4t) - \sin(4t) = 0 \]
\[
\sin(4t) = \cos(4t) \]

Divide by \( \cos(4t) \). Then
\[
\sin(4t)/\cos(4t) = 1
\]
Figure 1: The predator-prey system of squirrels and coyotes. The upper curve is the population of the squirrels at time $t$, while the lower curve tells the population of the coyotes at time $t$.

tan(4t) = 1 since \( \tan(4t) = \sin(4t)/\cos(4t) \).
But \( \tan(x) = 1 \) when \( x = \pi/4, 5\pi/4, 9\pi/4 \), etc. Hence the possible times are \( 4t = \pi/4, 4t = 5\pi/4, 4t = 9\pi/4 \), etc. Hence \( t = \pi/16, t = 5\pi/16, t = 9\pi/16 \) etc. But \( Y(\pi/16) = 300 + 10\sin(4\pi/16) + 10\cos(4\pi/16) = 314.14 \).
Similarly \( Y(3\pi/16) = 300 + 10\sin(20\pi/16) + 10\cos(20\pi/16) = 285.86 \), and \( Y(9\pi/16) = 314.14 \). Clearly the maximum number of squirrels is 314.14 thousand, and the minimum number of squirrels is 285.86 thousand.

3 Estimating the parameters from data

Suppose that we are given data for a predator-prey system and we wish to find the parameters of the model. The most important parameters to find are \( Y_e \), \( D_e \), \( a \), and \( b \).

(Step 1) \( Y_e \) should be the average value of \( Y(t) \). It is often best found by merely taking the average of the largest and smallest values, as long as a whole cycle is represented in the data. Alternatively, \( Y_e \) might be estimated by taking the average of all the values of \( Y \), but only over a whole number of cycles of oscillation. Similarly \( D_e \) should be the average of \( D(t) \).

(Step 2) Let \( y(t) = Y(t) - Y_e \) and \( d(t) = D(t) - D_e \) be the excess populations. Then we expect
\[
y'(t) = -ad(t) \\
d''(t) = by(t)
\]
Thus \( b \) should be the average of \( d''(t)/y(t) \), and \(-a \) should be the average.
value of \( g'(t)/d(t) \).

*Example.* Suppose we have the following data. (These data are inadequate for a good fit, but they illustrate the process on a small dataset. An actual dataset should include many more time values.)

\[
\begin{array}{ccc}
t & Y & D \\
0.0 & 27.00 & 5.90 \\
0.1 & 27.01 & 6.48 \\
0.2 & 26.25 & 6.96 \\
0.3 & 26.02 & 7.15 \\
0.4 & 23.77 & 6.97 \\
0.5 & 23.00 & 6.50 \\
0.6 & 22.99 & 5.92 \\
0.7 & 23.75 & 5.44 \\
0.8 & 24.98 & 5.25 \\
0.9 & 26.22 & 5.43 \\
1.0 & 27.00 & 5.43 \\
\end{array}
\]

We first seek to estimate the average value \( Y_c \) of \( Y \). The largest value is 27 and the smallest is 23, so we expect the typical value to be halfway between these; we pick \( Y_c = (23+27)/2 = 25 \). Similarly the largest value of \( D \) is 6.5 and the smallest value is 5.9, so we pick \( D_c = 6.2 \).

Now we find the excess populations. These are found by the formulas \( y(t) = Y(t) - Y_c = Y(t) - 25 \) and \( d(t) = D(t) - D_c = D(t) - 6.2 \). These formulas lead to tables for \( y \) and \( d \):

\[
\begin{array}{cccc}
t & Y & D & y & d \\
0.0 & 27.00 & 5.90 & 2.00 & -0.3 \\
0.1 & 27.01 & 6.48 & 2.01 & 0.28 \\
0.2 & 26.25 & 6.96 & 1.26 & 0.76 \\
0.3 & 26.02 & 7.15 & 0.02 & 0.95 \\
0.4 & 23.77 & 6.97 & -1.23 & 0.77 \\
0.5 & 23.00 & 6.50 & -2.00 & 0.30 \\
0.6 & 22.99 & 5.92 & -2.01 & -0.28 \\
0.7 & 23.75 & 5.44 & -1.25 & -0.76 \\
0.8 & 24.98 & 5.25 & -0.02 & -0.95 \\
0.9 & 26.22 & 5.43 & 1.22 & -0.77 \\
1.0 & 27.00 & 5.90 & 2.00 & -0.30 \\
\end{array}
\]

We need to estimate \( y' \) and \( d' \). Since we have yet no formula for \( y \) or \( d \), we cannot use the rules for differentiation. Instead, we must merely estimate the derivatives from the data present. When we have data at times \( t_0, t_1, t_2, \ldots, t_n \), an approximation is

\[
f'(t_i) \approx (f(t_{i+1}) - f(t_i))/(t_{i+1} - t_i).
\]

For example, \( y'(0.1) \approx (y(0.2) - y(0.1))/(0.2 - 0.1) = (1.25 - 2.01)/(0.2 - 0.1) = -7.6 \). We then also produce \( y'/d \) at each value of \( t \).
<table>
<thead>
<tr>
<th>t</th>
<th>Y</th>
<th>D</th>
<th>y</th>
<th>d</th>
<th>y'</th>
<th>y'/d</th>
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We expect from the model that \( y'(t) = -ad(t) \). Hence we expect \( y'(t)/d(t) = -a \). The column of \( y'/d \) shows that we are not getting a constant value. (If the time intervals were shorter, the values probably would be more nearly constant.) The best we can do here is to take the average of the values \( y'/d \). This average is -13.36612. This leads us to pick \( a = 13.36612 \). Similarly, since the model suggests \( d'(t) = by(t) \), we try to calculate \( d'(t) \). Then \( d'(t)/y(t) \) should equal \( b \).

When we do this to the given data set, we obtain

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<tr>
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<td>-0.95</td>
<td>1.8</td>
<td>-90</td>
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<td>2.00</td>
<td>-0.30</td>
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</table>

In this case the values of \( d'/y \) at \( t = 0.3 \) and \( t = 0.8 \) are clearly outliers, obtained merely because the value of \( y \) there was very close to 0 (and division by 0 is, of course, forbidden). Hence we ignore those values but take the average of the remaining values. This leads us to

\[
b = \frac{(2.9+2.388059701+1.52+3.821138211+2.9+2.388059701+1.52+3.852459016)}{8} = 2.66121458.
\]

Then \( \omega = \sqrt{ab} = 5.96 \).

We now know that our model should be

\[
Y(t) = 25 + C_1 \sin(5.96t) + C_2 \cos(5.96t)
\]

\[
D(t) = 6.2 + C_3 \sin(5.96t) + C_4 \cos(5.96t)
\]

Since \( Y(0) = 27 \) it follows \( 27 = 25 + C_2 \) so \( C_2 = 2 \). Since \( D(0) = 5.9 \), it follows \( 5.9 = 6.2 + C_4 \) so \( C_4 = -0.3 \). Now
\[ y(t) = C_1 \sin(5.96t) + 2 \cos(5.96t) \]
\[ d(t) = C_2 \cos(5.96t) - 0.3 \cos(5.96t) \]
\[ y'(t) = 5.96C_1 \cos(5.96t) - 11.92 \sin(5.96t) \]
But \[ y'(t) = -ad(t) = -13.36612(C_2 \sin(5.96t) - 0.3 \cos(5.96t)) \]
Hence \[ 5.96C_1 \cos(5.96t) - 11.92 \sin(5.96t) = -13.36612(C_2 \sin(5.96t) - 0.3 \cos(5.96t)) \]
We match the coefficients of the sine and cosine functions on each side of the equation. Then \[ 5.96C_1 = (-13.36612)(-0.3) \text{ and } -11.92 = -13.36612C_3. \] We conclude \( C_1 = 0.673 \) and \( C_3 = 0.892 \).

The model of these data is therefore
\[ Y(t) = 25 + 0.673 \sin(5.96t) + 2.00 \cos(5.96t) \]
\[ D(t) = 6.2 + 0.892 \sin(5.96t) - 0.3 \cos(5.96t) \]

4 Problems

1. Find the general solution:
   
   (a) \[ y''(t) + 25y(t) = 0 \]  
   (b) \[ y''(t) + 10y(t) = 0 \]  
   (c) \[ y'' + 2.135y = 0 \]  
   (d) \[ u'' = -7.396u \]
   
   Answers: (a) \( y = C_1 \sin(5t) + C_2 \cos(5t) \)  
   (b) \( y = C_1 \sin(\sqrt{10}t) + C_2 \cos(\sqrt{10}t) \)  
   (c) \( y = C_1 \sin(1.46116t) + C_2 \cos(1.46116t) \)  
   (d) \( u = C_1 \sin(2.719559t) + C_2 \cos(2.719559t) \)

2. Suppose \( y(t) \) satisfies \[ y''(t) + 9y(t) = 0, y(0) = 5, y'(0) = 12 \]. Find \( y(t) \).
   Answer: \( y(t) = 4 \sin(3t) + 5 \cos(3t) \)

3. Suppose \( v(t) \) satisfies \[ v''(t) + 16v(t) = 0, v(0) = -6, v'(0) = 20 \]. Find \( v(t) \).
   Answer: \( v(t) = 5 \sin(4t) - 6 \cos(4t) \)

4. Suppose \( u(t) \) satisfies \[ u''(t) + 11u(t) = 0, u(0) = 5, u'(0) = -7 \]. Find \( u(t) \).
   Answer: \( u(t) = -(7/\sqrt{11}) \sin(\sqrt{11}t) + 5 \cos(\sqrt{11}t) \)

5. Suppose \( y(t) \) satisfies \[ y'' + 17.3y = 0, y(0) = 1.2, y'(0) = -2.7 \].
   (a) Find \( y(t) \).
   (b) Find \( y(0.15) \).
   Answers: (a) \( y = -0.6491435 \sin(4.1593257t) + 1.2 \cos(4.1593257t) \)  
   (b) \( 0.594696 \)

6. Suppose \( u'(t) = -6u(t) \) and \( v'(t) = 3u(t) \).
   (a) Find the differential equation for \( u(t) \) in terms of its own derivatives.
   (b) Find the general solution for part (a).
   (c) Find \( v(t) \) in terms of the constants used in the answer for (b).
   (d) Suppose \( u(0) = 300 \) and \( u'(0) = 120 \). Find \( u(t) \) and \( v(t) \).
   Answers: (a) \( u'' + 18u = 0 \)
(b) \( u(t) = C_1 \sin(4.24264t) + C_2 \cos(4.24264t) \)
(c) \( v(t) = 0.70711C_2 \sin(4.24264t) - 0.70711C_1 \cos(4.24264t) \)
(d) \( u(t) = 28.284 \sin(4.24264t) + 300 \cos(4.24264t) \)
\( v(t) = 212.13 \sin(4.24264t) - 20.00 \cos(4.24264t) \)

7. Suppose \( u'(t) = -8v, \ v'(t) = 2u, \ u(0) = 160, \ (0) = 15 \).
(a) Find \( u(t) \) and \( v(t) \).
(b) Find \( u(3) \).
Answers: (a) \( u(t) = -30 \sin(4t) + 160 \cos(4t) \)
\( v(t) = 80 \sin(4t) + 15 \cos(4t) \) (b) 151

8. Suppose \( r'(t) = -4s(t) \) and \( s'(t) = 9r(t) \). Assume \( r(0) = 80 \) and \( s(0) = 6 \).
Find \( r(t) \) and \( s(t) \).
Answers: 
\( r(t) = -4 \sin(6t) + 80 \cos(6t) \),
\( s(t) = 120 \sin(6t) + 6 \cos(6t) \)

9. Suppose \( u'(t) = -5v(t) \) and \( v'(t) = 4u(t) \). Assume \( u(0) = 490 \) and \( v(0) = 55 \).
Find \( u(t) \) and \( v(t) \).
Answers: 
\( u(t) = -61.492 \sin(4.47214t) + 490 \cos(4.47214t) \)
\( v(t) = 438.27 \sin(4.47214t) + 55 \cos(4.47214t) \)

10. (a) Suppose that theory suggests \( y = ax \) for some constant \( a \). Given the following data, find the best estimate for \( a \) to two decimal places, and tell the resulting equation for \( y \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
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<tbody>
<tr>
<td>3.1</td>
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</tr>
<tr>
<td>3.2</td>
<td>15.1</td>
</tr>
<tr>
<td>3.3</td>
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<td>3.5</td>
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<td>17.1</td>
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<tr>
<td>4.0</td>
<td>18.6</td>
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</table>

(b) Suppose that, instead, theory suggests \( y = mx + b \) for some constants \( m \) and \( b \). Given the data in part (a), find the best (least squares) estimate for the equation, telling the parameters to two decimal places.
(c) Explain why the coefficients of \( x \) in the answers to (a) and (b) can be different.
Answers (a) \( 4.70 \) or \( y = 4.70x \) (b) \( y = 4.31x + 1.37 \)

11. Suppose that in a predator-prey system the prey consists of rabbits with population \( R(t) \) at time \( t \) and the predator consists of foxes with population \( F(t) \) at time \( t \) (in years). Suppose that on average the rabbit population is 50,000 rabbits, while the fox population is on average 2000 foxes. Let \( r(t) \) and \( f(t) \) be the excess rabbit and fox populations. Suppose that
\[ r'(t) = -45 f(t) \text{ and } f'(t) = 20 r(t) \]
Suppose that initially the rabbit population is 51,000 rabbits while the fox population is 1900 foxes.
(a) Find formulas for the rabbit and fox populations at time \( t \).
(b) What is the calculated rabbit population 2 years after the start?
(c) What is the calculated fox population 2 years after the start?
(d) What is the instantaneous rate of the change of the rabbit population 2 years after the start?
(e) Is the rabbit population increasing or decreasing 2 years after the start?
(f) What is the instantaneous rate of the change of the fox population 2 years after the start?
(g) Is the fox population increasing or decreasing 2 years after the start?
(h) What is the first time \( t \) for which the rabbit population is a maximum?
(i) What is the maximum rabbit population?
(j) What is the first time \( t \) for which the rabbit population is a minimum?
(k) What is the minimum rabbit population?

Answers:
(a) \( R(t) = 50000 + 150 \sin(30t) + 1000 \cos(30t) \)
(b) 49002 rabbits
(c) 1892 foxes
(d) \( R'(2) = 4858 \text{ rabbits/year} \)
(e) increasing
(f) \( F'(2) = -19964 \text{ foxes/year} \)
(g) decreasing
(h) 0.004963 years
(i) 51011 rabbits
(j) 0.10968 years
(k) 48989 rabbits