Generalized Jackson Networks
(Approximate Decomposition Methods)

• The arrival and departure processes are approximated by renewal process and each station is analyzed individually as GI/G/m queue.

• The performance measures (E[N], E[W]) are approximated by 4 parameters \{C_{a_{0,j}}, C_{S_{j}}, \lambda_{0,j}, \mu_{j}\}. (Don’t need the exact distribution)


Single-class GI/G/1 OQNS

Notations:

• $n$ = number of internal stations in the network. For each $j$, $j = 1, \ldots, n$:

• $\lambda_{0,j} = \text{expected external arrival rate at station } j$
  
  $[\lambda_{0,j} = 1/E[(A_{0,j})]]$,

• $C_{a_{0,j}}^2 = \text{squared coefficient of variation (scv) or variability of external interarrival time at station } j$
  
  $\{C_{a_{0,j}}^2 = [\text{Var}(A_{0,j})]/[E(A_{0,j})^2]\}$,

• $\mu_j = \text{expected service rate at station } i$ $[\mu_j = 1/E(S_j)]$

• $C_{s_{0,j}}^2 = \text{scv or variability of service time at station } j$
  
  $\{C_{s_{0,j}}^2 = [\text{Var}(S_j)]/[E(S_j)^2]\}$.
Notations (cont)

- For each pair \((i,j)\), \(i = 1,\ldots,n, j = 1,\ldots,n\):
  \[ p_{ij} = \text{probability of a job going to station } j \text{ after completing service at station } i. \]
  *We assume no immediate feedback, \(p_{ii} = 0\)

- For \(n\) nodes, the input consists of \(n^2 + 4n\) numbers

- The complete decomposition is essentially described in 3 steps
  - Step 1. Analysis of interaction between stations of the networks,
  - Step 2. Evaluation of performance measures at each station,
  - Step 3. Evaluation of performance measures for the whole network.
Step 1:

- Determine two parameters for each stations $j$:

  (i) the expected arrival rate $\lambda_j = 1/E[A_j]$ where $A_j$ is the interarrival time at the station $j$.

  (ii) the squared coefficient of variation (scv) or variability of interarrival time, $C_{a_i}^2 = \text{Var}(A_j)/E(A_j)^2$.

For $\lambda_j$:

- $\lambda_j$ can be obtained from the traffic rate equations:

  $\lambda_j = \lambda_{0_j} + \sum_{i=1}^{n} \lambda_{ij}$ for $j = 1, \ldots, n$

  where $\lambda_{ij} = p_{ij} \lambda_i$ is the expected arrival rate at station $j$ from station $i$. We also get $\rho_j = \lambda_j/\mu_j$, $0 \leq \rho_j \leq 1$
Step 1: (cont.)

- The expected (external) departure rate to station 0 from station $j$ is given by $\lambda_{j0} = \lambda_j \left(1 - \sum_{i=1}^{n} p_{ji}\right)$.
- Throughput $\lambda_0 = \sum_{j=1}^{n} \lambda_{0j}$ or $\sum_{j=1}^{n} \lambda_{j0}$
- The expected number of visits $v_j = E(K_j) = \lambda_j/\lambda_0$.

The s.c.v. of arrival time can be approximated by Traffic variability equation (Whitt(1983b))

$$C^2_{a_j} = w_j \sum_{i=0}^{n} \frac{\lambda_{ij}}{\lambda_j} C^2_{a_j} + 1 - w_j$$

Where $w_j = \frac{1}{1 + 4(1 - \rho_j)^2 (x_j - 1)}$ and $x_j = \frac{1}{\sum_{i=0}^{n} \left(\frac{\lambda_{ij}}{\lambda_j}\right)^2}$
Step 1: (cont.)

\[ C_{a_{ji}}^2 = \text{the interarrival time variability at station } j \text{ from station } i. \]

\[ C_{d_{ji}}^2 = \text{the departure time variability at station } j \text{ from } i. \]

\[ C_{a_{ji}}^2 = C_{d_{ji}}^2 = p_{ji} C_{d_{j}}^2 + 1 - p_{ji} \]

\[ C_{d_{j}}^2 \text{ is interdeparture time variability from station } j. \]

\[ C_{d_{j}}^2 = \rho_j^2 C_{s_{j}}^2 + (1 - \rho_j^2) C_{a_{j}}^2 \]

Note that if the arrival time and service processes are Poisson (i.e., \( C_{a_{j}}^2 = C_{s_{j}}^2 = 1 \)), then \( C_{d_{j}}^2 \) is exact and leads to \( C_{d_{j}}^2 = 1. \)

Note also that if \( \rho_j \rightarrow 1 \), then we obtain \( C_{d_{j}}^2 \rightarrow C_{s_{j}}^2 \)

On the other hand if \( \rho_j \rightarrow 0 \), then we obtain \( C_{d_{j}}^2 \rightarrow C_{a_{j}}^2 \)
Step 2:

- Find the expected waiting time by using Kraemer & Lagenbach-Belz formula [modified by Whitt(1983a)]

$$E(W_j) = \frac{\rho_j (C_{a_j}^2 + C_{S_j}^2) g(\rho_j, C_{a_j}^2, C_{S_j}^2)}{2\mu_j (1 - \rho_j)} = \frac{(C_{a_j}^2 + C_{S_j}^2) g(\rho_j, C_{a_j}^2, C_{S_j}^2)}{2} E[W]_{M/M/1}$$

where

$$g(\rho_j, C_{a_j}^2, C_{S_j}^2) = \begin{cases} 
\exp\left\{ -2(1 - \rho_j)(1 - C_{a_j}^2)^2 \right\} & \text{if } C_{a_j}^2 < 1 \\
\frac{-3\rho_j (C_{a_j}^2 + C_{S_j}^2)}{\rho_j (C_{a_j}^2 + C_{S_j}^2)} & \text{if } C_{a_j}^2 \geq 1 \\
1 & \text{if } C_{a_j}^2 \geq 1 
\end{cases}$$

Step 3:

- Expected lead time for an arbitrary job (waiting times + service times)

$$E(T) = \sum_{j=1}^{n} v_j (E(W_j) + E(S_j))$$
Single-class GI/G/m OQNS with probabilistic routing

More general case where there are \( m \) identical machines at each station \( i \).

\[ C_{a_j}^2 = \alpha_j + \sum_{i=1}^{n} C_{a_j}^2 \beta_{ij} \quad \text{for} \quad j = 1, \ldots, n \]

Where

\[ \alpha_j = 1 + w_j \left\{ (p_{0j} C_{a_0}^2 - 1) + \sum_{i=1}^{n} p_{ij} [(1 - q_{ij}) + q_{ij} \rho_i^2 x_i] \right\} \]

\[ \beta_{ij} = w_j p_{ij} q_{ij} (1 - \rho_i^2) \]

\[ w_j = \frac{1}{1 + 4(1 - \rho_j)^2 (x_j - 1)} \]

\[ q_{ij} = \frac{\lambda_i}{\lambda_j} p_{ij} \left( q_{ij} = \frac{\lambda_{ij}}{\lambda_j}, p_{ij} = \frac{\lambda_{ij}}{\lambda_i} \right) \quad \text{and} \quad x_i = 1 + \frac{(\max\{C_{S_i}^2, 0.2\} - 1)}{\sqrt{m_i}} \]
Single-class $GI/G/m$ OQNS with probabilistic routing (cont.)

- Then the approximation of expected waiting time is similar to GI/G/1 case:

$$E(W_j) = \frac{\left(C_{a_j}^2 + C_{S_j}^2\right)}{2} E(W_j(M/M/m_j))$$
General Job Shop (text sec. 7.7)

Rely on approximations for
\[ E[N] = \sum_{i=1}^{m} E[N_i] \text{ and } E[T] = \sum_{i=1}^{m} \nu_i E[T_i] \]

These can be found from a sample path analysis of each machine center using
s.c.v. of interarrival times: \( C_{a_i}^2 \)
s.c.v. of service times: \( C_{s_i}^2 \)

Probability distribution of service times
\[ F_{S_i}(t) = P[S_i \leq t] \]
Single Machine, Poisson External Arrivals

From Chapter 3, if there is one server in each m/c center, the mean time spent at m/c center $i$ by an arbitrary job can be approximated by using $4$ parameters, $\lambda_i, \mu_i, C_{a_i}^2, C_{S_i}^2$ as shown in equation 3.142, 3.143 and 3.144 in the text.

If $C_{a_i}^2 \leq 2$ use

$$\hat{T}_i = \left\{ \frac{\rho_i^2 (1+C_{S_i}^2)}{1+\rho_i^2 C_{S_i}^2} \right\} \left\{ \frac{(C_{a_i}^2 + \rho_i^2 C_{S_i}^2)}{2\lambda (1-\rho_i)} \right\} + E[S_i]$$

(3.142)

or

$$\hat{T}_i = \left\{ \frac{\rho_i^2 (1+C_{S_i}^2)}{2-\rho_i + \rho_i C_{S_i}^2} \right\} \left\{ \frac{\rho_i (2-\rho_i) C_{a_i}^2 + \rho_i^2 C_{S_i}^2}{2\lambda (1-\rho_i)} \right\} + E[S_i]$$

(3.143)

If $C_{a_i}^2 \leq 1$ use

$$\hat{T}_i = \frac{\rho_i (2-\rho_i) C_{a_i}^2 + \rho_i^2 C_{S_i}^2}{2\lambda (1-\rho_i)} + E[S_i]$$

(3.144)
Approximating Parameters

Four parameters $\lambda_i, \mu_i, C^2_{a_i}, C^2_{s_i}$ can be approximated by solving linear system of equations below:

$$\lambda_i = \lambda \gamma_i + \sum_{j=1}^{n} \lambda_j p_{ji}, i = 1,\ldots,m$$

$$C^2_{d_i} = 1 + (1 - \rho_i^2)(C^2_{a_i} - 1) + \frac{\rho_i^2}{C_{s_i}} (C^2_{s_i} - 1), i = 1,\ldots,m$$

$$C^2_{a_i} = \frac{1}{\lambda_i} \sum_{j=1; j \neq i}^{m} \lambda_j p_{ji} [p_{ji} C^2_{d_j} + (1 - p_{ji})] + \frac{\lambda \gamma_i}{\lambda_i} [\gamma_i C^2_{a_i} + (1 - \gamma_i)], i = 1,\ldots,m$$

The first equation find $\lambda_i$. The second and third equation find s.c.v of arrival and departure time. Note that the first and second equations are the same as in Whitt(1983) while the third equation is different.

We’ll consider two extremes of job routing diversity.
Symmetric Job Shop

A job leaving m/c center $i$ is equally likely to go to any other m/c next:

$$p_{ij} = 1/m, j \neq i; \quad p_{i0} = 1/m \quad \text{(leaves system)}$$

Then

$$\lambda_i = \lambda, i = 1, \ldots, m;$$

$$C_{a_i}^2 \to 1 \quad \text{as} \quad m \to \infty$$

So we can approximate each m/c center as M/G/1.

Then, $\hat{W}(\lambda_i, \mu_i, C_{a_i}^2, C_{s_i}^2)$ can be replaced by mean waiting time in M/G/1 queue.

$$E[W_i] = \frac{\lambda E[S_i^2]}{2(1 - \rho_i)} = \frac{\rho_i^2 (1 + C_{s_i}^2)}{[2\lambda_i(1 - \rho_i)]}$$
# Symmetric Job Shop

## TABLE 7.1 MEAN FLOW TIME IN TWO-MACHINE–CENTER SYMMETRIC JOB SHOP

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda$</th>
<th>Simulation 95% C.I.</th>
<th>Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$G1$-arrival</td>
<td>$M$-arrival</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>2.98 ± 0.12</td>
<td>3.04</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>3.97 ± 0.16</td>
<td>3.92</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>5.94 ± 0.27</td>
<td>5.76</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>2.82 ± 0.10</td>
<td>2.81</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>3.62 ± 0.14</td>
<td>3.53</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>5.52 ± 0.24</td>
<td>5.16</td>
</tr>
<tr>
<td>5</td>
<td>0.4</td>
<td>2.72 ± 0.09</td>
<td>2.68</td>
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<tr>
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<td>0.6</td>
<td>3.40 ± 0.13</td>
<td>3.32</td>
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<tr>
<td></td>
<td>0.8</td>
<td>5.52 ± 0.24</td>
<td>4.87</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.4</td>
<td>2.61 ± 0.09</td>
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<tr>
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<td>0.6</td>
<td>3.41 ± 0.13</td>
<td>3.33</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>6.23 ± 0.28</td>
<td>5.28</td>
</tr>
</tbody>
</table>

## TABLE 7.2 MEAN FLOW TIME IN FOUR-MACHINE–CENTER SYMMETRIC JOB SHOP

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda$</th>
<th>Simulation 95% C.I.</th>
<th>Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$G1$-arrival</td>
<td>$M$-arrival</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>6.11 ± 0.23</td>
<td>6.07</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>8.16 ± 0.34</td>
<td>7.85</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>11.90 ± 0.47</td>
<td>11.53</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>5.64 ± 0.21</td>
<td>5.63</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>7.43 ± 0.27</td>
<td>7.11</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>11.21 ± 0.48</td>
<td>10.45</td>
</tr>
<tr>
<td>5</td>
<td>0.4</td>
<td>5.43 ± 0.19</td>
<td>5.37</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>7.22 ± 0.28</td>
<td>6.71</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>10.98 ± 0.48</td>
<td>9.96</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.4</td>
<td>5.21 ± 0.19</td>
<td>5.28</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>7.19 ± 0.29</td>
<td>6.76</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>12.97 ± 0.58</td>
<td>10.95</td>
</tr>
</tbody>
</table>
Uniform Flow Job Shop

All jobs visit the same sequence of machine centers. Then
\[
\lambda_i = \lambda, i = 1, \ldots, m;
\]

but \( C_{a_i}^2 \) is unaffected by \( m \)

In this case, \( M/G/1 \) is a bad approximation, but \( GI/G/1 \) works fine.

The waiting time of \( GI/G/1 \) queue can be approximated by using equation 3.142, 3.143 or 3.144 above and the lower and upper bound is

\[
\frac{\rho_i(C_{a_i}^2 - 1 + \rho_i) + \rho_i^2C_{s_i}^2}{2\lambda(1 - \rho_i)} \leq \hat{W}(\lambda_i, \mu_i, C_{a_i}^2, C_{s_i}^2) \leq \frac{\rho_i(2 - \rho_i)C_{a_i}^2 + \rho_i^2C_{s_i}^2}{2\lambda(1 - \rho_i)}
\]
### Uniform-Flow Shop

#### TABLE 7.4 MEAN FLOW TIME IN TWO-MACHINE-CENTER UNIFORM-FLOW SHOP

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda$</th>
<th>Simulation 95% C.I.</th>
<th>Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$GI$-arrival</td>
<td>$M$-arrival</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>$3.06 \pm 0.10$</td>
<td>3.04</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>$3.87 \pm 0.14$</td>
<td>3.92</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>$5.82 \pm 0.25$</td>
<td>5.76</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>$2.70 \pm 0.08$</td>
<td>2.79</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>$3.53 \pm 0.12$</td>
<td>3.48</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>$5.18 \pm 0.21$</td>
<td>4.97</td>
</tr>
<tr>
<td>5</td>
<td>0.4</td>
<td>$2.56 \pm 0.06$</td>
<td>2.66</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>$3.26 \pm 0.10$</td>
<td>3.24</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>$4.68 \pm 0.19$</td>
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</tr>
<tr>
<td>$\infty$</td>
<td>0.4</td>
<td>$2.33 \pm 0.04$</td>
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<td>$2.76 \pm 0.08$</td>
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<tr>
<td></td>
<td>0.8</td>
<td>$4.03 \pm 0.16$</td>
<td>4.60</td>
</tr>
</tbody>
</table>

#### TABLE 7.5 MEAN FLOW TIME IN FOUR-MACHINE-CENTER UNIFORM FLOW SHOP

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda$</th>
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<th>Approximation</th>
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</thead>
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<tr>
<td></td>
<td></td>
<td>$GI$-arrival</td>
<td>$M$-arrival</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>$6.14 \pm 0.18$</td>
<td>6.07</td>
</tr>
<tr>
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<td>$7.77 \pm 0.25$</td>
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<td>$11.34 \pm 0.38$</td>
<td>11.53</td>
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<tr>
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<td>0.4</td>
<td>$5.37 \pm 0.13$</td>
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<td>$6.74 \pm 0.19$</td>
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<td>$9.83 \pm 0.37$</td>
<td>9.14</td>
</tr>
<tr>
<td>5</td>
<td>0.4</td>
<td>$4.93 \pm 0.10$</td>
<td>5.14</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>$5.99 \pm 0.15$</td>
<td>5.84</td>
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<td>$8.44 \pm 0.30$</td>
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</tr>
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<td>$\infty$</td>
<td>0.4</td>
<td>$4.32 \pm 0.04$</td>
<td>4.96</td>
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<tr>
<td></td>
<td>0.6</td>
<td>$4.72 \pm 0.08$</td>
<td>5.48</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>$6.13 \pm 0.22$</td>
<td>6.70</td>
</tr>
</tbody>
</table>
Job Routing Diversity

Assume high work load: \( \rho_i \approx 1, i = 1, \ldots, m \)
and machine centers each with same number of machines,
same service time distribution and same utilization. Also,

\[ v_i = 1, i = 1, \ldots, m \]

Q: What job routing minimizes mean flow time?

A1: If \( C_a^2 \geq C_s^2 \) and \( C_s^2 \leq 1 \) then uniform-flow job routing minimizes mean flow time;

A2: If \( C_a^2 \leq C_s^2 \) and \( C_s^2 \geq 1 \) then symmetric job routing minimizes mean flow time.
Variance of Flow Time

The mean flow time is useful information but not sufficient for setting due dates. The two distributions below have the same mean, but if the due date is set as shown, tardy jobs are a lot more likely under distribution B!
Flow Time of an Aggregate Job

Pretend we are following the progress of an arbitrary tagged job. If $K_i$ is the number of times it visits m/c center $i$, then its flow time is ($K_i$ is the r.v. for which $v_i$ is the mean)

$$T = \sum_{i=1}^{m} \sum_{j=1}^{K_i} T_{ij},$$

where $T_{ij}$ is the time spent in $i$ on $j^{th}$ visit

Given the values of each $K_i$, the expected flow time is

$$E[T|K_i,i=1,...,m] = \sum_{i=1}^{m} K_i E[T_i]$$

and then, “unconditioning”,

$$E[T] = \sum_{i=1}^{m} E[K_i] E[T_i] = \sum_{i=1}^{m} v_i E[T_i]$$
Variance

In a similar way, we can find the variance of $T$ conditional upon $K_i$ and then uncondition.

Assume $\{T_{ij}, i = 1, \ldots, m; j = 1, \ldots\}$ and $\{K_i, i = 1, \ldots, m\}$ are all independent and that, for each $i$, $\{T_{ij}, j = 1, \ldots\}$ are identically distributed.

Similar to conditional expectation, there is a relation for conditional variance:

$$\text{Var}[X] = E_Y \left[ \text{Var} \left( X | Y \right) \right] + \text{Var}_Y \left[ E \left( X | Y \right) \right]$$
Using Conditional Variance

Since $T$ depends on $\{K_1, K_2, \ldots, K_m\}$, we can say

$$\text{Var}(T) = E_{K_1, K_2, \ldots, K_m} \left[ \text{Var}(T \mid K_1, \ldots, K_m) \right]$$

$$+ \text{Var}_{K_1, K_2, \ldots, K_m} \left[ E(T \mid K_1, \ldots, K_m) \right]$$

The first term equals

$$E_{K_1, K_2, \ldots, K_m} \left[ \text{Var} \left( \sum_{i=1}^{m} \sum_{j=1}^{K_i} T_{ij} \mid K_1, \ldots, K_m \right) \right] = E_{K_1, K_2, \ldots, K_m} \left[ \sum_{i=1}^{m} K_i \text{Var}(T_i) \right]$$

$$= \sum_{i=1}^{m} E(K_i) \text{Var}(T_i) = \sum_{i=1}^{m} \nu_i \text{Var}(T_i)$$
Using Conditional Variance

The second term is

$$\text{Var}_{K_1, K_2, \ldots, K_m} \left[ \mathbb{E} \left( \sum_{i=1}^{m} \sum_{j=1}^{K_i} T_{ij} \mid K_1, \ldots, K_m \right) \right] = \text{Var}_{K_1, K_2, \ldots, K_m} \left[ \sum_{i=1}^{m} K_i \mathbb{E}(T_i) \right]$$

$$= \sum_{i=1}^{m} \text{Var}[K_i \mathbb{E}(T_i)] + 2 \sum_{1 \leq i < j \leq m} \text{Cov}[K_i \mathbb{E}(T_i), K_j \mathbb{E}(T_j)]$$

$$= \sum_{i=1}^{m} \mathbb{E}(T_i)^2 \text{Var}[K_i] + 2 \sum_{1 \leq i < j \leq m} \mathbb{E}(T_i) \mathbb{E}(T_j) \text{Cov}[K_i, K_j]$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \text{Cov}[K_i, K_j] \mathbb{E}(T_i) \mathbb{E}(T_j)$$
Variance

The resulting formula for the variance is:

\[ \text{Var}[T] = \sum_{i=1}^{m} v_i \text{Var}[T_i] + \sum_{i=1}^{m} \sum_{j=1}^{m} \text{Cov}[K_i, K_j]E[T_i]E[T_j] \]

If arrivals to each m/c center are approximately Poisson, we can find \( \text{Var}[T_i] \) from the M/G/1 transform equation (3.73), p. 61.

\[ E[T_i] = -\frac{d\tilde{F}_{T_i}(s)}{ds}\bigg|_{s=0}, \quad E[T_i^2] = \frac{d^2\tilde{F}_{T_i}(s)}{ds^2}\bigg|_{s=0}, \quad \text{Var}[T_i] = E[T_i^2] - E[T_i]^2 \]

But we still need \( \text{Cov}(K_i, K_j) \).
Markov Chain Model

Think of the tagged job as following a Markov chain with states 1,…,m for the machine centers and absorbing state 0 representing having left the job shop.

The transition matrix is

$$
\begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 - \sum_{j=1}^{m} p_{1j} & 0 & p_{12} & \ldots & p_{1m} \\
1 - \sum_{j=1}^{m} p_{2j} & p_{21} & 0 & \ldots & p_{2m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 - \sum_{j=1}^{m} p_{mj} & p_{m1} & p_{m2} & \ldots & 0
\end{bmatrix}
$$

$K_i$ is the number of times this M.C. visits state $i$ before being absorbed;

let $K_{ji}$ be the number of visits given $X_0 = j$. 
\[ K_{ji} = \begin{cases} \delta_{ji}, & \text{with probability } 1 - \sum_{l=1}^{m} p_{jl} \\ K_{li} + \delta_{ji}, & \text{with probability } p_{jl}, \ l = 1, \ldots, m \end{cases} \] (7.83)

Where \( \delta_{ji} = 1 \) if \( j = i \) and 0 otherwise.

\[ K_{ji} K_{jr} = \begin{cases} \delta_{ji} \delta_{jr}, & \text{with probability } 1 - \sum_{l=1}^{m} p_{jl} \\ (K_{li} + \delta_{ji})(K_{lr} + \delta_{jr}), & \text{with probability } p_{jl}, \ l = 1, \ldots, m \end{cases} \] (7.84)
Expectations

Take expectation of (7.83), we get

\[
\left( E\left[ K_{ji} \right] \right)_{j,i=1,...,m} = (I - P)^{-1}
\]

and take expectation of (7.84),

\[
E[K_{ji} K_{jr}] = E[K_{jr}]E[K_{ri}] + E[K_{ji}]E[K_{ir}], \quad i \neq r
\]

and

\[
E[K_{ji}^2] = E[K_{ji}]\{2E[K_{ii}] - 1\}
\]
Because

\[ E[K_i K_r] = \sum_{i=1}^{m} \gamma_j E[K_{ji}] E[K_{jr}] \]

and

\[ E[K_i^2] = \sum_{j=1}^{m} \gamma_j E[K_{ji}^2] \], we obtain

\[ E[K_i K_r] = E[K_i] E[K_{ir}] + E[K_r] E[K_{ri}], i \neq r \]

and

\[ E[K_i^2] = E[K_i] \{2E[K_{ii}] - 1\} \]

where

\[ E[K_i] = \sum_{j=1}^{m} \gamma_j E[K_{ji}] \]

Therefore

\[ \text{Cov}[K_i, K_j] = \begin{cases} E[K_i] E[K_{ij}] + E[K_j] E[K_{ji}] - E[K_i] E[K_j], & i \neq j \\ E[K_i] \{2E[K_{ii}] - E[K_i] - 1\}, & i = j \end{cases} \]