MATH 373: HOMEWORK 5
“INTERPOLATION I”
FALL 2013

NOTE: For each homework assignment observe the following guidelines:

• Include a cover page.

• Always clearly label all plots (title, x-label, y-label, and legend).

• Use the subplot command when comparing 2 or more plots to make comparisons easier and to save paper.

1. Consider the function \( f(x) = \sin(x) \).

   (a) Construct the Lagrange form of the interpolating polynomial for \( f \) passing through the points
   \[
   (0, \sin(0)), \quad \left( \frac{\pi}{4}, \sin \left( \frac{\pi}{4} \right) \right), \quad \left( \frac{\pi}{2}, \sin \left( \frac{\pi}{2} \right) \right).
   \]

   (b) Plot the polynomial from part (a) and the function \( f(x) = \sin(x) \) on the same graph. Use \( x \) in the range \([0, \pi/2]\).

   (c) Plot the difference between the polynomial from part (a) and the function \( f(x) = \sin(x) \). Use \( x \) in the range \([0, \pi/2]\).

   (d) Use the polynomial from part (a) to estimate both \( \sin(\pi/3) \) and \( \sin(\pi/6) \). What is the error in each approximation.

   (e) Establish the theoretical error bound for using the polynomial found in part (a) to approximate \( \sin(\pi/3) \). Compare the theoretical error bound to the error found in part (d).

2. Consider the data set

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

   (a) Show that the polynomials \( p(x) = x^3 + 2x^2 - 3x + 1 \) and \( q(x) = \frac{1}{8} x^4 + \frac{3}{4} x^3 + \frac{15}{8} x^2 - \frac{11}{4} x + 1 \) both interpolate all of the data.

   (b) Why does this not contradict the theorem on the existence and uniqueness of polynomial interpolation?

3. Suppose that \( f \in C^2(x_0, x_1) \). Derive the following bound on the error due to linear interpolation of \( f \):

   \[
   |f(x) - p_1(x)| \leq \frac{h^2}{8} \max_{x_0 \leq x \leq x_1} |f''(x)|,
   \]

   where \( h = x_1 - x_0 \).
4. The following table gives the viscosity, in mili-Pascal-seconds (centipoises) of sulfuric acid as a function of concentration, in mass percent.

<table>
<thead>
<tr>
<th>Concentration</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Viscosity</td>
<td>0.89</td>
<td>1.40</td>
<td>2.51</td>
<td>5.37</td>
<td>17.4</td>
<td>24.2</td>
</tr>
</tbody>
</table>

(a) Determine the polynomial of degree at most five which interpolates this data.
(b) The viscosity of sulfuric acid with a 5% concentration is 1.01 and with a 10% concentration is 1.12. Use these values to assess the accuracy of the interpolating polynomial.

5. Construct the divided difference table of the following data set, and then write out the Newton form of the interpolating polynomial.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>2</td>
<td>-1</td>
<td>4</td>
</tr>
</tbody>
</table>

6. Construct the divided difference table of the following data set, and then write out the Newton form of the interpolating polynomial.

<table>
<thead>
<tr>
<th>x</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>3</td>
<td>-1</td>
<td>-3</td>
<td>1</td>
</tr>
</tbody>
</table>

7. **SOURCE CODE:**

Let \( x_0, x_1, \ldots, x_n \) be distinct points with corresponding data values \( f_0, f_1, \ldots, f_n \). Let \( P(x) \) be the unique polynomial of degree \( n \) which interpolates the data (i.e., \( P(x_i) = f_i \) for each \( i = 0, \ldots, n \)). Write the following MATLAB routines:

(a) \( F = \text{DivDiff}(x,f,n) \) — a function to compute the divided difference table \( F \).
(b) \( P = \text{Horners}(x,F,xbar) \) — a function that uses the \( x \)-location of the interpolation points, \( x \), and the divided difference table \( F \) to evaluate \( P(x) \) at the point (or list of points) \( xbar \) using Horner’s method.

8. **APPLICATION PROBLEM:**

Consider the function

\[
    f(x) = \frac{1}{1 + 25x^2}.
\]

(a) Let \( x_i \) be 11 equally spaced points on \([-1, 1] \):

\[
    x_i = -1 + i/5, \quad \text{where} \quad i = 0, 1, 2, \ldots, 10.
\]
i. Use your code from the previous problem to construct the divided differences with these \( x_i \)'s.

ii. Use your Horner’s method code to evaluate the interpolating polynomial \( P(x) \) at all the points \( z_k \):

\[
z = \text{linspace}(-1,1,500);
\]

iii. Make a single MATLAB plot that contains: \((z,P(z)\), \((z,f(z))\), and the data points \((x_i,f(x_i))\) for \( i = 0,1,2,\ldots10 \). Make the \( P(z) \) plot a solid line ('b-'), the \( f(z) \) a dashed line ('r--'), and the data points stars ('k*').

(b) Now, let \( x_i \) be 11 unequally spaced points on \([-1,1]\):

\[
x_i = \cos \left[ \frac{\pi (i + 0.5)}{11} \right], \quad \text{where} \quad i = 0,1,2,\ldots,10.
\]

Repeat steps (i), (ii), and (iii) for these unequally spaced data points.

(c) Explain what you observe with these two choices of the \( x_i \)'s.

9. APPLICATION PROBLEM:

Consider the following values for the sound speed, \( a \) (meters/second), in water as a function of temperature \( T \) (Celcius).

<table>
<thead>
<tr>
<th>( T ) (Celcius)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a ) (meters/second)</td>
<td>1402</td>
<td>1447</td>
<td>1482</td>
<td>1509</td>
<td>1529</td>
<td>1542</td>
<td>1551</td>
<td>1553</td>
<td>1554</td>
<td>1550</td>
<td>1543</td>
</tr>
</tbody>
</table>

(a) Apply your divided differences code to obtain the coefficients of the Newton form of the interpolating polynomial.

(b) Use your Horner’s method code to determine the sound speed in water when \( T = 34, T = 68, T = 86, \) and \( T = 91 \).