Solutions to Homework #5

2.8
In all the case \( a \) is the minimal solution of \( \phi(a) = a \) in \((0, 1]\).

(b) We have:

\[
a = p_0 + p_1a + p_3a^3 \iff 4a^3 - 9a + 5 = 0 \iff (a - 1)(4a^2 + 4a - 5) = 0.
\]

Hence \( a = \frac{\sqrt{5} - 1}{2} \).

(d) We have: \( \phi(s) = \frac{1-q}{1-qs} \). Hence,

\[
\alpha = \frac{1-q}{1-q\alpha} \iff 1 - q + q\alpha^2 = 0,
\]

which implies that \( \alpha = \frac{1-q}{q} \) if \( q \geq 1/2 \) and \( \alpha = 1 \) otherwise.

2.9
In general,

\[
\P(X_{n+1} = 0 | X_n \neq 0) = \frac{\P(X_{n+1} = 0 \cap X_n \neq 0)}{\P(X_n \neq 0)} = \frac{\P(X_{n+1} = 0) - \P(X_n = 0)}{1 - \P(X_n = 0)}
\]

\[= \frac{\alpha_{n+1} - \alpha_n}{1 - \alpha_n} = \frac{\phi(\alpha_n) - \alpha_n}{1 - \alpha_n}.
\]

(a) We have:

\[
\P(X_2 = 0 | X_1 \neq 0) = \frac{\P(X_2 = 0) - \P(X_1 = 0)}{1 - \P(X_1 = 0)} = \frac{\phi(p_0) - p_0}{1 - p_0} = \frac{p_0 + p_1p_0 + p_3p_0^3 - p_0}{1 - p_0}
\]

\[= \frac{p_1p_0 + p_3p_0^3}{1 - p_0}.
\]

(b) We have:

\[
\P(X_3 = 0 | X_2 \neq 0) = \frac{\P(X_3 = 0) - \P(X_2 = 0)}{1 - \P(X_2 = 0)} = \frac{\phi(\phi(p_0)) - \phi(p_0)}{1 - \phi(p_0)}.
\]

Since \( \phi(s) = p_0 + p_1s + p_3s^3 \) it is in principle possible to compute it even without calculators.
2.11 Let $\mu := \sum_{n=1}^{\infty} np_n$. Using the notation of the textbook, $\mathbb{P}(Y_1 = 0) = p_0q$ and $\mathbb{P}(Y_1 = n) = p_nq + p_{n-1}(1-q)$, because the parent particle counts if it doesn’t die out. Hence,

$$\tilde{\mu} := \mathbb{E}(Y_1) = \sum_{n=1}^{\infty} n(p_nq + p_{n-1}(1-q)) = 1 + \mu - q.$$ 

Therefore, the extinction is certain if $q \geq \mu$.

2.13

(a) Using the notation of the textbook,

$$\mathbb{P}(Y_1 = 0) = 1 - q, \quad \mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = 2) = \frac{q}{2}.$$ 

Hence,

$$\mu := \mathbb{E}(Y_1) = \frac{q}{2} + 2\frac{q}{2} = \frac{3q}{2}.$$ 

Therefore, the extinction is certain if $q \leq 2/3$.

(b) Using the notation of the textbook, we must compute $1 - \alpha(4) = 1 - [\alpha(1)]^4$. We have:

$$\alpha(1) = 1 - q + \frac{q}{2} \alpha(1) + \frac{q}{2} \alpha(2) = 1 - q + \frac{q}{2} \alpha(1) + \frac{q}{2} [\alpha(1)]^2.$$ 

Therefore (because $\alpha(1)$ is the minimal root),

$$\alpha(1) = \frac{2(1 - q)}{q},$$ 

and $1 - \alpha(4) = 1 - 16\left(\frac{1-q}{q}\right)^4$.

2.14

(a) Assume that $\phi'(a) \geq 1$. Then, since $\phi'(s)$ is monotone increasing,

$$1 - a = \phi(1) - \phi(a) = \int_a^1 \phi'(s)ds > \phi'(a)(1 - a) \geq 1 - a.$$ 

This contradiction establishes the fact that $\phi'(a) < 1$.

(b) Denote $\rho := \phi'(a)$. Using the mean value theorem for derivatives and the fact that $\alpha_n < \alpha$ (see p. 56 in the text),

$$\alpha - \alpha_{n+1} = \phi(\alpha) - \phi(\alpha_n) \leq \rho(\alpha - \alpha_n).$$ 

Using induction, $\alpha - \alpha_n \leq \rho^n(\alpha - \alpha_0) = \alpha \rho^n$. 

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(c) Let $\mathcal{E} := \{\lim_{n \to \infty} X_n = 0\}$ denote the event of extinction. Then,

$$
\mathbb{P}(\mathcal{E}|X_n \neq 0) = \frac{\mathbb{P}(\mathcal{E} \cap X_n \neq 0)}{\mathbb{P}(X_n \neq 0)} = \frac{\mathbb{P}(\mathcal{E}) - \mathbb{P}(\mathcal{E} \cap X_n = 0)}{1 - \mathbb{P}(X_n = 0)} = \frac{\alpha - \alpha_n}{1 - \alpha_n} \leq \frac{\alpha - \alpha_n}{1 - \alpha},
$$

where in the last inequality we used the fact that $\alpha_n < \alpha$ (see p. 56 in the text). Thus

$$
\mathbb{P}(\mathcal{E}|X_n \neq 0) \leq \frac{\alpha \rho^{n-1}}{1 - \alpha}.
$$