Solutions to Homework #3

1.15 (a)
For any state \( j \neq i \), we have:

\[
\tau(j) = \mathbb{E}_i \left( \sum_{n=0}^{T-1} I(X_n = j) \right) = \mathbb{E}_i \left( \sum_{n=0}^{\infty} I(X_n = k, n \leq T - 1) \right)
\]

\[
= \sum_{n=0}^{\infty} \mathbb{P}_i(X_n = j, T > n) = \sum_{n=1}^{\infty} \mathbb{P}_i(X_n = j, T > n),
\]

where in the last step we used the fact \( \mathbb{P}_i(X_0 = j) = 0 \) for \( j \neq i \).

Hence, using the Markov property,

\[
\tau P(j) = \sum_{k \in \Omega} \tau(k) P(k, j) = \sum_{n=0}^{\infty} \sum_{k \in \Omega} \mathbb{P}_i(X_n = k, T > n) P(k, j)
\]

\[
= \sum_{n=0}^{\infty} \mathbb{P}_i(X_{n+1} = j, T > n + 1) = \sum_{n=1}^{\infty} \mathbb{P}_i(X_n = j, T > n) = \tau(j).
\]

On the other hand, \( \tau(i) = 1 \) and

\[
\tau P(i) = \sum_{k \in \Omega} \tau(k) P(k, i) = \sum_{n=0}^{\infty} \sum_{k \in \Omega} \mathbb{P}_i(X_n = k, T > n) P(k, i)
\]

\[
= \sum_{n=0}^{\infty} \mathbb{P}_i(T = n + 1) = \sum_{n=1}^{\infty} \mathbb{P}_i(T = n) = 1.
\]

Thus \( \tau = \tau P \).

1.15 (b)
Notice that the events \( \{X_n = j\}, j \in S \), form a disjoint partition of \( S \). In particular, \( \sum_{j \in S} I(X_n = j) = 1 \).

Therefore, we have:

\[
\sum_{j \in \Omega} \tau(j) = \sum_{j \in \Omega} \mathbb{E}_i \left( \sum_{n=0}^{T-1} I(X_n = j) \right) = \mathbb{E}_i \left( \sum_{n=0}^{T-1} \sum_{j \in \Omega} I(X_n = j) \right)
\]

\[
= \mathbb{E}_i \left( \sum_{n=0}^{T-1} 1 \right) = \mathbb{E}_i(T).
\]

1.15 (c)
The claim is implied by the results in (a) and (b) since \( \pi \) is the unique (up to a multiplication factor) left-eigenvector of \( P \) corresponding to the eigenvalue 1. More precisely, it follows from (a)
that \( r = c\pi \) for a constant \( c \neq 0 \) (the constant cannot be zero, for instance, because \( r(i) = 1 \)). It then follows from (b) that \( c = \mathbb{E}_i(T) \). Since \( r(i) = 1 \), we conclude that
\[
1 = r(i) = \mathbb{E}_i(T) \cdot \pi(i),
\]
and hence \( \pi(i) = 1/\mathbb{E}_i(T) \), as desired. In fact, in addition, we can also see now that
\[
\pi(j) = \mathbb{E}_i(T) \cdot \pi(j) = \pi(j)/\pi(i),
\]
the result that we have stated and justified heuristically in the class.

1.16
Identify for a moment state 0 with \( N \). Then the event \( \{X_{T_N} = k\} \) for \( k = 1, \ldots, N-1 \) is the (disjoint) union of the following two events:
\[
A := \{X_{T_N} = k, \text{the walk has been visited to } k + 1 \}
\]
but never crossed the bond \((k, k + 1)\) before time \( T_n \}\]
and
\[
B := \{X_{T_N} = k, \text{the walk has been visited to } k - 1 \}
\]
but never crossed the bond \((k, k - 1)\) before time \( T_n \}\].
Consider first the event \( A \). Identify now
\[
N - j \text{ with } -j \text{ for } j = 1, \ldots, N - k.
\]
In particular, \( k + 1 = N - (N - (k + 1)) \) is now identified with \( N - (k + 1) \). Cut the circle at point \( k \) and unfold it into the interval \([- (N - k), k]\]. Let \( Q_i \) be the probability distribution of a simple symmetric nearest-neighbor random walk \( Y_n \) on this interval, starting at \( Y_0 = i \). Let \( \tau_k \) be the first hitting time of site \( k \). Then, using the Markov property,
\[
\mathbb{P}(A) = Q_0(\tau_{-(N-k-1)} < \tau_k) \times Q_{-(N-k-1)}(\tau_k < \tau_{-(N-k)}).
\]
Using the solution to the gambler’s ruin problem, we obtain:
\[
\mathbb{P}(A) = \frac{k}{N - 1} \cdot \frac{1}{N} = \frac{k}{N(N - 1)}.
\]
Similar arguments show that
\[
\mathbb{P}(B) = \frac{N - k}{N(N - 1)},
\]
and hence \( \mathbb{P}(X_{T_N} = k) = \mathbb{P}(A) + \mathbb{P}(B) = \frac{1}{N-1} \). If you want to consider the degenerate case \( k = N - 1 \) separately, you can use the above argument for \( k = 1, \ldots, N - 2 \) and then observe that
\[
\mathbb{P}(X_{T_N} = N - 1) = 1 - \sum_{k=1}^{N-2} \mathbb{P}(X_{T_N} = k).
\]
1.17 (a) Let \( \Omega = \{1, \ldots, N\} \). At first step the walk will move from 1 elsewhere besides 1. We have \( \mathbb{P}(i, 1) = \frac{1}{N-1} \) for any \( i \in \Omega, \ i \neq 1 \). Thus, for any \( m \in \mathbb{N} \),
\[
\mathbb{P}(T = m + 1) = \left( \frac{N - 2}{N - 1} \right)^m \cdot \frac{1}{N - 1}.
\]
This is a geometric distribution with parameter $\frac{1}{N-1}$. In particular, for any $m \in \mathbb{N}$,

$$\mathbb{P}(T > m) = \sum_{n=m}^{\infty} \mathbb{P}(T = n + 1) = \frac{1}{N-1} \cdot \sum_{n=m}^{\infty} \left(\frac{N-2}{N-1}\right)^{n-1} = \left(\frac{N-2}{N-1}\right)^{m-1}$$

Thus, by virtue of formula (1.13) in the textbook, we have

$$\mathbb{E}(T) = \sum_{m=0}^{\infty} \mathbb{P}(T > m) = 2 + \sum_{m=2}^{\infty} \left(\frac{N-2}{N-1}\right)^{m-1} = 2 + \frac{N-2}{N-1} \cdot \frac{1}{1 - \frac{N-2}{N-1}} = N.$$

By a symmetry argument, $\pi(j) = \frac{1}{N}$ for any $j \in \Omega$, and hence (1.11) holds in this particular instance.

**1.17 (b)** In fact, the proof in (a) tells us that

$$\mathbb{P}(T = m + 1 | X_0 = 1) = 1 + \mathbb{P}(T = m | X_0 = i)$$

for any $i \neq 1$. In words, $T - 1$ under $\{X_0 = 1\}$ is distributed as $T$ under $\{X_0 = i\}$. In particular,

$$\mathbb{E}(T | X_0 = 2) = \mathbb{E}(T | X_0 = 1) - 1 = N - 1.$$

**1.17 (c)**

Let

$$R_n = \{i \in \Omega : X_k = i \text{ for some } k = 0, \ldots, n\}$$

be the range of the random walk at time $n$. Let $r_n = |R_n|$ be the cardinality (number of elements) of the set $R_n$. Finally, let $\tau_1 = 0$ and for $n = 1, \ldots, N - 1$,

$$\tau_{n+1} = \inf\{k > \tau_n : r_k = n + 1\}.$$

Thus $\tau_n$ is the first time $k \geq 0$, such that $r_k = n$. We thus want to compute $\mathbb{E}(\tau_N)$. The rest of the argument is very similar to that of part (a).

Notice that $g_n := \tau_{n+1} - \tau_n$ are geometric random variables with

$$\mathbb{P}(g_n = m) = \left(\frac{n-1}{N-1}\right)^{m-1} \cdot \frac{N-n}{N-1}, \quad m \in \mathbb{N}.$$

Hence

$$\mathbb{E}(g_n) = \sum_{m=0}^{\infty} \mathbb{P}(g_n > m) = 1 + \sum_{m=1}^{\infty} \left(\frac{n-1}{N-1}\right)^{m-1} = 1 + \frac{n-1}{N-1} \cdot \frac{1}{1 - \frac{n-1}{N-1}} = \frac{N-1}{N-n},$$

and

$$\mathbb{E}(\tau_N) = \sum_{k=1}^{N-1} \mathbb{E}(g_k) = (N-1) \sum_{k=1}^{N-1} \frac{1}{k} \sim N \log N, \quad \text{as } N \to \infty.$$

**1.18** Combining together the results mentioned in the textbook about the stationary distribution
distribution of this Markov chain and the result of Exercise 1.15(c), we obtain that $E(T) = 52!$ in seconds. Hence, in years,

$$E(T) = \frac{52!}{60^2 \cdot 24 \cdot 365}.$$ 

1.19

Let $X_n$, $n \in \mathbb{N}$, be the results of $n$-th tossing of the coin. Define $Y_n$ as follows:

$$Y_n = \max\{ j \in [0, n] : X_n = \cdots = X_{n-j+1} = \text{HEAD} \}.$$ 

Thus $Y_n$ is the length of the current run of HEAD’s, which might be anything between zero (when $X_n = \text{T}A\text{IL}$) and $n$ (when no TAIL occurred in the first $n$ trials). We are interested in

$$T = \inf\{ n \in \mathbb{N} : Y_n = 4 \},$$

and can consider the Markov chain $Y_n$ only until time $T$. In other words, we can consider a Markov chain $Y_n$ in the state space $\{0, 1, 2, 3, 4\}$, with $Y_0 = 0$, absorbing state at 4, and transition kernel

$$P = \begin{bmatrix}
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 \\
1/2 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

If $u_i = E(T|Y_0 = i)$, then

$$u_0 = 1 + \frac{u_0}{2} + \frac{u_1}{2},$$

$$u_1 = 1 + \frac{u_0}{2} + \frac{u_2}{2},$$

$$u_2 = 1 + \frac{u_0}{2} + \frac{u_3}{2},$$

$$u_3 = 1 + \frac{u_0}{2}.$$ 

Substituting the last row into the third one, we obtain the following system of equations:

$$u_0 = 1 + \frac{u_0}{2} + \frac{u_1}{2},$$

$$u_1 = 1 + \frac{u_0}{2} + \frac{u_2}{2},$$

$$u_2 = \frac{3}{2} + \frac{3u_0}{4}.$$ 

Substituting the last row into the second one, we obtain that

$$u_0 = 1 + \frac{u_0}{2} + \frac{u_1}{2},$$

$$u_1 = \frac{7}{4} + \frac{7u_0}{8}.$$ 

Therefore,

$$u_0 = 1 + \frac{u_0}{2} + \frac{u_1}{2} = \frac{15}{8} + \frac{15u_0}{16}.$$ 

It follows from the last identity that $u_0 = 30$. 

4