Test #1 (solutions to the sample)

1 [20 Points].

(a) Solve Exercise 1.5 in the textbook.

(b) A fair die is thrown repeatedly. Let $X_n$ denote the sum of the first $n$ throws. Find

$$\lim_{n \to \infty} \mathbb{P}(X_n \text{ is a multiple of } 13).$$

Solution:

(a) There are three communication classes. $S_1 = \{0, 1\}$ and $S_2 = \{2, 4\}$ are recurrent, and $S_3 = \{3, 5\}$ is transient. Let $\pi = \{\pi_0, \pi_1\}$ be the stationary distribution of the chain within the recurrent communication class. Then $\pi_0 = 0.5\pi + 0.7\pi_1$, and hence $\pi_0 = 3/5\pi_1$. Thus $\pi_0 = 3/8$ and $\pi_1 = 5/8$. Since $P$ restricted to $S_1$ is aperiodic, it follows that

$$\lim_{n \to \infty} P^n(0, 0) = \frac{1}{\pi_0} = \frac{8}{3}.$$

Furthermore, using the notation of Section 1.5 (see pp. 29-30), we obtain

$$S = \begin{bmatrix} 0.25 & 0.25 & 0 & 0.25 \\ 0 & 0.2 & 0 & 0.2 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0.25 \\ 0.2 & 0.4 \end{bmatrix}, \quad I - Q = \begin{bmatrix} 1 & -0.25 \\ -0.2 & 0.6 \end{bmatrix},$$

$$M = (I - Q)^{-1} = \frac{1}{11} \begin{bmatrix} 12 & 5 \\ 4 & 20 \end{bmatrix}, \quad A = MS = \frac{1}{11} \begin{bmatrix} 3 & 4 & 0 & 4 \\ 1 & 5 & 0 & 5 \end{bmatrix},$$

and

$$\lim_{n \to \infty} P^n(5, 0) = \alpha(5, 0) + \alpha(5, 1) = A(2, 1) + A(2, 2) = \frac{6}{11}.$$

(b) Let $Y_n$ be the remainder of $X_n$ after dividing by 13. That is $X_n - Y_n$ is a multiple of 13 and $0 \leq Y_n \leq 12$. Then $Y_n$ is a Markov chain with transition kernel $P$ defined as follows:

$$P(i, j) = 1/6 \text{ if } i + 1 \leq j \leq i + 6 \text{ or } i + 1 - 13 \leq j \leq i + 6 - 13,$$

and $P(i, j) = 0$ otherwise. In words, if the states from 0 to 12 are arranged around a circle in the natural order clockwise, then $P(i, j) = 1/6$ as long as $j$ is located within the distance between 1 to 6 from $i$, moving clockwise. It is not hard to check that the chain is an aperiodic and irreducible and that the uniform distribution on $\{0, \ldots, 12\}$ is invariant for it. Hence, the limit is equal to $\frac{1}{13}$. 

1
2 [20 Points].

(a) Solve Exercise 2.6 (a) in the textbook.
(b) Solve Exercise 2.6 (b) in the textbook.
(b) Solve Exercise 2.6 (c) in the textbook.

Solution:

(a) \( p(i, i + 1) = p \) and \( p(i, 0) = 1 - p \).

(b) We have:

\[
\pi_0 = (1 - p) \sum_{i=0}^{\infty} \pi_i = 1 - p,
\]

and, for \( i \in \mathbb{N} \),

\[
\pi_i = p \pi_{i-1}.
\]

Using induction, \( \pi_i = p^i (1 - p) \).

(c) Since the chain is positive recurrent, \( \mathbb{E}(T_k) < \infty \). Once the state \( k - 1 \) reached, in the next step the chain moves to \( k \) with probability \( p \) and returns to zero with probability \( q \). Therefore,

\[
\mathbb{E}(T_k) = \mathbb{E}(T_{k-1}) + \mathbb{E}(T_k - T_{k-1}) = \mathbb{E}(T_{k-1}) + p \cdot 1 + q \cdot (1 + \mathbb{E}(T_k))
\]

\[
= \mathbb{E}(T_{k-1}) + 1 + q \mathbb{E}(T_k).
\]

Hence,

\[
\mathbb{E}(T_k) = \frac{1}{p} + \frac{1}{p} \mathbb{E}(T_{k-1}).
\]

Iterating the last identity, we obtain

\[
\mathbb{E}(T_k) = \frac{1}{p} + \frac{1}{p} \mathbb{E}(T_{k-1}) = \frac{1}{p} + \frac{1}{p} \left( \frac{1}{p} + \frac{1}{p} \mathbb{E}(T_{k-2}) \right) = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^2} \mathbb{E}(T_{k-2})
\]

\[
= \ldots = \sum_{j=1}^{k-1} p^{-j} + \frac{1}{p^{k-1}} \mathbb{E}(T_1).
\]

Since \( T_1 \) is a geometric random variable with parameter \( p \) (i.e., \( \mathbb{P}(T_1 = m) = q^{m-1} p \) for \( m \geq 1 \), we have \( \mathbb{E}(T_1) = p^{-1} \). Thus

\[
\mathbb{E}(T_k) = \sum_{j=1}^{k} p^{-j} = \frac{1}{p} \frac{1 - p^{-k}}{1 - p^{-1}} = \frac{1}{q} (p^{-k} - 1).
\]
2 [20 Points]. [20 points]

(a) Solve Exercise 2.7 (a) in the textbook.

(b) Solve Exercise 2.7 (c) in the textbook.

Solution:

To establish whether or not the underlying Markov chain is transient, it is convenient to use the criterion stated as the Fact on p. 50. Let $z = 0$ and assume that $\alpha(x)$ defined by Equations (2.2)–(2.4) on p. 49 exists. In particular, for $x > 0$ we have:

$$\alpha(x) = \sum_{y=0}^{\infty} p(x, y) \alpha(y) = p(x, 0) \alpha(0) + p(x, x + 1) \alpha(x + 1)$$

$$= p(x, 0) + p(x, x + 1) \alpha(x + 1),$$

and hence

$$\alpha(x + 1) = \alpha(x) \cdot \frac{1}{p(x, x + 1)} - \frac{p(x, 0)}{p(x, x + 1)}.$$

Using induction (iterating the above inequality), it is not hard to verify that

$$\alpha(2) = \frac{\alpha(1)}{p(1, 2)} - \frac{p(1, 0)}{p(1, 2)}$$

and, for $x > 1$,

$$\alpha(x + 1) = -\frac{p(x, 0)}{p(x, x + 1)} - \sum_{y=1}^{x-1} p(y, y + 1) \prod_{t=y+1}^{x} \frac{1}{p(t, t + 1)} + \alpha(1) \cdot \prod_{t=1}^{x} \frac{1}{p(t, t + 1)}$$

Using the probabilistic meaning of $\alpha(x) = \mathbb{P}_x(X_n = 0$ for some $n \geq 0)$, one conclude that $\alpha(x)$ is a non-increasing function of $x$ given by

$$\alpha(x) = 1 - \prod_{t=x}^{\infty} p(t, t + 1). \quad (1)$$

(a) It follows from (1) that $\alpha(x) \equiv 1$ and hence the chain is recurrent. To see whether the chain is positive- or null-recurrent, let $T = \inf\{k > 0 : X_k = 0\}$ and compute

$$\mathbb{E}_0(T) = \sum_{n=2}^{\infty} n \mathbb{P}_0(T = n) = \sum_{n=2}^{\infty} n \prod_{t=0}^{n-2} p(t, t + 1) \cdot p(n - 1, 0)$$

$$= \sum_{n=2}^{\infty} n \prod_{t=0}^{n-2} \frac{t + 1}{t + 2} \cdot \frac{n}{n + 1} = \sum_{n=2}^{\infty} \frac{n}{n + 1} = +\infty.$$ 

Thus the chain is null-recurrent.
(c) It follows from (1) that

$$\alpha(x) = 1 - \prod_{t=x}^{\infty} \left(1 - \frac{1}{t^2 + 2}\right),$$

and hence \(\lim_{x \to \infty} \alpha(x) = 0\). Thus the chain is transient.

4 [20 Points].

(a) Solve Exercise 2.8 (a) in the textbook.

(b) Solve Exercise 2.8 (c) in the textbook.

Solution:

(a) \(a\) is the minimal solution of \(\phi(a) = a\) in \((0, 1]\). We have:

$$a = p_0 + p_1 a + p_2 a^2 \iff 35a^2 - 60a + 25 = 0$$

Hence \(a = 5/7\).

(c) We have, \(\mu = 0.05 + 0.01(2 + 3 + 6 + 13) = 0.29 < 1\). Thus the extinction is certain.

5 [20 Points]. Solve Exercise 2.16 in the textbook.

Solution:

(a) By the definition of the invariant measure,

$$\pi(n) = \sum_{m=0}^{\infty} \pi(m)p(n,m) = \sum_{m=0}^{\infty} \pi(m)p_{n-m}, \quad n \geq 1. \quad (2)$$

Since \(p_{n-m} = 0\) if \(n - m > 1\), it follows that \(\pi(n) = \sum_{m=n-1}^{\infty} \pi(m)p_{n-m}\).

(b) Let \(Y\) be a random variable valued in the set of non-negative integers, such that \(\mathbb{P}(Y = n) = q_n, \ n \geq 0\). Then

$$\mathbb{E}(Y) = \sum_{n=0}^{\infty} q_n n = \sum_{n=0}^{\infty} p_{1-n} n = -\sum_{n=0}^{\infty} p_{1-n} (1 - n) + \sum_{n=0}^{\infty} p_{1-n}$$

$$= -\sum_k p_k k + 1 = -\mu + 1 > 0,$$

since it is given that \(\mu < 0\). To conclude that the equation \(\alpha = \mathbb{E}(\alpha^Y)\) has a root within the interval \((0, 1)\), one can use the same argument as for the super-critical branching processes.

(c) This involves a guess. The correct answer is \(\pi(n) = c\alpha^n\) which can be validated directly using (2) together with the fact that \(\sum_{n=0}^{\infty} \pi(n) = 1\). The latter also implies \(c = (1 - \alpha)^{-1}\).