1 [20 Points].
(a) Let \((X_n)_{n\geq 0}\) be a discrete-time Markov chain with state space \(\Omega = \{1, 2, 3, 4\}\) and transition matrix

\[
P = \begin{pmatrix}
0 & 1/2 & 0 & 1/2 \\
1/2 & 0 & 1/2 & 0 \\
0 & 1/2 & 0 & 1/2 \\
1/2 & 0 & 1/2 & 0
\end{pmatrix}
\]

Recall cyclic partition \(\Omega = C_0 \cup C_1 \cdots \cup C_{d-1}\) introduced in the class, such that \(P(i, j) > 0\) only if \(i \in C_k\) and \(j \in C_{k+1}\) for some \(k\) (we identify here \(C_0\) with \(C_d\)). It can be shown that \((X_n)_{n\geq 0}\) induces partition \(\Omega = C_0 \cup C_1\) with \(C_0 = \{1, 3\}\) and \(C_1 = \{2, 4\}\) (consider this as a given fact, you do not have to verify it). In particular, \(d = 2\). Find \(\lim_{n\to\infty} P^{2n+r}(i,j)\) for \(r = 0, 1\) and \(i, j \in \{1, 2\}\).

(b) Solve problem 2.4 from the textbook.

2 [20 Points]. Solve problem 1.12 from the textbook.

3 [20 Points]. Three players, say A, B, and C, participate in a tournament with the following rules. First, A and B play a game. The winner then plays with C. The winner in the second game meets then the one who lost in the first game, and so on until one of the players wins two games in row. The player who succeeds first to win two games in row is announced as the winner of the tournament. Assume that each player wins in any given game with probability 0.5, independently of the previous results in tournament.

(a) Compute the probability that A wins the tournament.

(b) Compute the probability that C wins the tournament.

(c) Compute the expected numbers of games in the tournament.

4 [20 Points]. Let \((X_n)_{n\geq 0}\) be a discrete-time Markov chain with state space \(\{1, 2\}\) and transition matrix

\[
P = \begin{pmatrix}
1/3 & 2/3 \\
3/4 & 1/4
\end{pmatrix}
\]

Further, let \(H\) be a \(2 \times 2\) matrix given by

\[
H(i, j) = P(i, j)j, \quad i, j \in \{1, 2\}.
\]

(a) Show by induction that \(E(X_1X_2 \cdots X_n; X_n = j|X_0 = i) = H^n(i,j)\), where \(i, j \in \{1, 2\}\).

(b) Compute \(\lim_{n\to\infty} \frac{1}{n} \log E(X_1X_2 \cdots X_n|X_0 = i)\) for \(i \in \{1, 2\}\).

5 [20 points] Solve problem 2.7 (b) from the textbook.

6 [bonus] Solve problem 2.15 from the textbook.