Bonus homework problems

Submit no later than during the first week after Thanksgiving. You can hand in solutions to any number of problems listed below at any time before than. The assignment will be constantly (but maybe not very regularly) updated until Thanksgiving.

1. Show that at least one of the communication classes of a finite-state Markov chain must be recurrent.

2. Solve \( f(n) = 1 + pf(n+1) + qf(n-1) \) for arbitrary \( p \in (0, 1) \) and \( q = 1 - p \).

3. Verify the claim made in Example 4 of Section 1.3.2. (p. 21 of the textbook) that the simple random walk on a connected graph has period \( d = 2 \) if and only if the graph is bipartite.

4* (Hard). Prove the following ultimate extension of the previous result: An irreducible Markov chain has period \( d \) if and only if the state space \( S \) can be partitioned into \( d \) disjoint sets \( S_1, \ldots, S_d \) such that, setting \( S_{d+1} = S_1 \), we have

\[
p(i, j) > 0 \text{ only if } i \in S_k, j \in S_{k+1} \text{ for some } k = 1, \ldots, d.
\]

5. Verify that if \( P^m > 0 \) for some \( m \in \mathbb{N} \), then

   1. \( P \) is irreducible.
   2. \( P^n > 0 \) for all \( n > m \).
   3. \( d = 1 \).

6. In the setting of Problem 4 above, consider a Markov chain with the state space \( S_k \) and transition kernel \( P_k \) defined by \( P_k(i, j) := P^d(i, j) \) for \( i, j \in S_k \). Show that this Markov chain is irreducible and aperiodic.

7. In the setting of Problems 4 and 6 above, let \( \pi_k \) be the invariant distribution for \( P_k \), that is \( \pi_k \) is the unique distribution over \( S_k \) such that \( \pi_k = \pi_k P_k \).

   1. Show that \( \pi := \frac{1}{d} \sum_{k=1}^{d} \pi_k \) is the invariant distribution for \( P \).
   2. Show that for any \( i = 1, \ldots, d \) and \( k = 1, \ldots, d \), we have

\[
\pi_i P^k = \begin{cases} 
\pi_{i+k} & \text{if } i + k \leq d, \\
\pi_{i+k-d} & \text{if } i + k > d
\end{cases}
\]

In other words, if the initial distribution of the original Markov chain \( X_n \) is the invariant measure of \( P_i \), then its distribution after \( k \) steps is the invariant distribution of \( P_m \), where \( m \) equals \( i + k \mod d \).
The following problem is a “final accord” in the proof of the LLN which we discussed in the class. Fix a state $i$ of an irreducible Markov chain $X_n$ with transition kernel $P$ on a finite state space $S = \{1, \ldots, N\}$. Let $f: S \to \mathbb{R}$ be an arbitrary function. Assume that $X_0 = i$ and let $\tau_n$ be the times of successive visits to $i$:

$$\tau_0 = 0 \quad \text{and} \quad \tau_k = \inf\{n > \tau_{k-1} : X_n = i\} \quad \text{for} \quad k \in \mathbb{N}.$$ 

Let $\eta_n$ be the number of visits to $i$ during the time interval $[1, n]$, that is $\eta_n$ is the unique non-negative integer such that

$$\tau_{\eta_n} \leq n < \tau_{\eta_n+1}.$$ 

The goal of this exercise is to prove that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=\tau_{\eta_n}}^{n} f(X_k) = 0, \quad \text{a.s.} \quad (1)$$

To accomplish this task;

(a) Observe that $f$ is bounded, let say $|f(i)| < M$ for some $M > 0$ and all $i \in S$. Conclude that in order to prove (1) it suffices to show that

$$\lim_{n \to \infty} \frac{n - \tau_{\eta_n}}{n} = 0 \quad \text{a.s.} \quad (2)$$

(b) Observe that by definition of the limit, (3) is equivalent to the claim that

$$P_i(n - \tau_{\eta_n} > n\varepsilon \text{ i.o.}) = 0, \quad \forall \varepsilon > 0. \quad (3)$$

where the abbreviation i.o. stands for “infinitely often”. Using the Borel-Cantelli lemma (see for instance Wikipedia on-line) conclude that (3) is equivalent to the claim that

$$\sum_{n=1}^{\infty} P_i(n - \tau_{\eta_n} > n\varepsilon) < \infty, \quad \forall \varepsilon > 0. \quad (4)$$

(c) Justify the following calculation:

$$\sum_{n=1}^{\infty} P_i(n - \tau_{\eta_n} > n\varepsilon) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} P_i(n - \tau_{\eta_n} > n\varepsilon, \eta_n = k) \leq \sum_{n=1}^{\infty} \sum_{k=0}^{n} P_i(\tau_{k+1} - \tau_k > n\varepsilon, \eta_n = k) \leq \sum_{n=1}^{\infty} \sum_{k=0}^{n} P_i(\tau_{k+1} - \tau_k > n\varepsilon) = \sum_{n=1}^{\infty} (n + 1) P_i(\tau_1 > n\varepsilon) \quad (5)$$

(d) Use (5) and the result in Problem 1.7 to verify (4).