Introduction to uncertainty analysis and error propagation

Every measurement has uncertainty because of the limited precision of instruments, errors in the instruments, user error, etc. When a physical quantity is computed from measurements of other quantities, uncertainty in the measured quantities will result in uncertainty in the computed quantity. For example, if you measure the volume \( V \) of a box with length \( L \), width \( B \), and height \( H \), then uncertainty in \( L \), \( B \), and \( H \) will propagate into uncertainty in \( V \). The question is how to compute the uncertainty in the volume.

Suppose that you use a meter stick with millimeter divisions to find a length of 0.5 m, a width of 0.3 m, and a height of 0.1 m. The uncertainty in each measurement could be estimated as 0.5 mm, or half the smallest unit on the meter stick; in other words, anything falling between two divisions would be difficult to quantify. A logical approach to estimate the volume’s uncertainty \( \Delta V \) is to compute it as

\[
\Delta V = (L + \Delta L)(B + \Delta B)(H + \Delta H) - LBH,
\]

or the difference between the actual volume and the volume of a box whose length, width, and height are overestimated by the uncertainties \( \Delta L \), \( \Delta B \), and \( \Delta H \), respectively. In this example, the uncertainty would be \(1.2 \times 10^{-4} \text{ m}^3\). However, this approach is somewhat unfair because it assumes the worst case: Each length is overestimated (or underestimated) by 0.5 mm. In reality, some lengths will be underestimated, and some will be overestimated. Therefore, the first-order uncertainty in a quantity \( y \) that depends on \( N \) quantities \( x_1, x_2, x_3, \ldots, x_N \), with uncertainties \( \Delta x_1, \Delta x_2, \Delta x_3, \ldots \Delta x_N \) is often computed with

\[
(\Delta y)^2 = \sum_{i=1}^{N} \left( \frac{\partial y}{\partial x_i} \Delta x_i \right)^2.
\]

For the box example, in which \( V = LBH \), the uncertainty in the volume is given by

\[
(\Delta V)^2 = \left( \frac{\partial V}{\partial L} \Delta L \right)^2 + \left( \frac{\partial V}{\partial B} \Delta B \right)^2 + \left( \frac{\partial V}{\partial H} \Delta H \right)^2
= (BH\Delta L)^2 + (LH\Delta B)^2 + (LB\Delta H)^2
= 2.3 \times 10^{-10} \text{ m}^6 + 6.3 \times 10^{-10} \text{ m}^6 + 56.3 \times 10^{-10} \text{ m}^6,
\]

so that \( \Delta V = 8 \times 10^{-5} \text{ m}^3 \), which is smaller than that from the first method. This analysis identifies the main contributors to the uncertainty. In this case, most of the uncertainty in the volume comes from uncertainty in the height. In fact, even if the length and width were measured perfectly (i.e., \( \Delta L = \Delta B = 0 \)), the uncertainty in volume would decrease by only 6%. Therefore, if you had extra money to spend to improve the experiment, you would spend it on measuring the height more accurately. Another way to present this
analysis is to consider the relative uncertainty \( \Delta V/V \). In this example and others (but not all), the analytical expression is fairly simple:

\[
\left( \frac{\Delta V}{V} \right)^2 = \left( \frac{\Delta L}{L} \right)^2 + \left( \frac{\Delta B}{B} \right)^2 + \left( \frac{\Delta H}{H} \right)^2 = (0.001)^2 + (0.002)^2 + (0.005)^2
\]

The relative uncertainty is then \( \Delta V/V = 0.6\% \). I often find this normalized form easier to interpret than the one with units.

A slightly more complex example illustrates the difference between sensitivity and uncertainty. Suppose that you measure the volume of a cylinder by measuring the diameter \( D \) and the height \( H \). Because the volume is \( V = \pi D^2 H/4 \), the volume is more sensitive to the diameter than the height. This sensitivity is reflected in the formula for the relative uncertainty in the volume:

\[
\left( \frac{\Delta V}{V} \right)^2 = \left( 2 \frac{\Delta D}{D} \right)^2 + \left( \frac{\Delta H}{H} \right)^2.
\]

The factor of 2 in the formula arises because the volume depends on the square of the diameter. If the relative uncertainties in the diameter and height are equal, then the diameter will contribute more to the uncertainty than the height. However, if the uncertainty in the height is more than twice the uncertainty in the diameter, the height will contribute more to the uncertainty.

More sophisticated error analyses distinguish between bias and random error. I encourage students who work with data (i.e., everyone) to seek further information on error and uncertainty analysis. For more, try the book by Taylor, *An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements*. 