Note in what follows that an AR(p) process is "invertible" by definition.

a) \( \phi(z) = 1 + 0.2z - 0.48z^2 = (1 + 0.8z)(1 - 0.6z) \)
\( \phi(z) = 0 \) when \( z = -\frac{0.8}{0.6} < -1 \) and \( z = \frac{1}{0.6} > 1 \).
Both roots outside the unit circle \( \implies \) process is causal.

b) \( \phi(z) = 1 + 1.9z + 0.88z^2 = (1 + 1.1z)(1 + 0.8z) \implies \) root \( -\frac{1}{1.1} = -0.91 \) inside unit circle
\( \implies \) process is NOT causal.

\[ \theta(z) = 1 + 0.2z + 0.7z^2 \implies \] roots are \( z_{1,2} = -0.2 \pm \frac{\sqrt{2.76}}{1.4} \)
\( |z_1| = |z_2| = \frac{\sqrt{0.04 + 2.76}}{(1.4)^2} = 1.429 > 1 \implies \) process is invertible.

\[ \phi(z) = 1 + 0.6z^2 \implies \] roots \( z_{1,2} = \pm \frac{\sqrt{10.16}}{1.2} = \frac{5}{1.2} \implies \) CAUSAL
\[ \theta(z) = 1 + 1.2z \implies \text{root } z = -\frac{1}{1.2} = -\frac{5}{6} \implies \text{NOT invertible}. \]

c) \( \phi(z) = 1 + 0.9z^2 \implies \) roots \( z_1 = z_2 = -\frac{\sqrt{0.81}}{0.9} = -1 \implies \) is CAUSAL.

d) \( \phi(z) = (1 + 0.9z)^2 \implies \) roots \( z_1 = z_2 = -\frac{1}{0.9} < -1 \implies \) is CAUSAL.

\( \theta(z) = 1 - 0.4z + 0.4z^2 = (1 - 0.2z)^2 \implies \) roots \( z_{1,2} = \frac{1}{0.2} = 5 \implies \) is INVERTIBLE.
**Problem 2:** $X_t \sim ARMA(2,4) \Rightarrow X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \Theta_1 W_{t-1} + \Theta_2 W_{t-2} + \Theta_3 W_{t-3} + \Theta_4 W_{t-4}$

Without loss of generality, let's assume $E(X_t) = 0$.

a) Causal representation: $X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j}$ and our task is to identify the weights as a function of $\{\phi_1, \phi_2, \ldots, \phi_p, \Theta_1, \ldots, \Theta_q\}$.

For this, we just equate the coefficients of $z^i$ in the representation:

$$(\psi_0 + \psi_1 z + \psi_2 z^2 + \ldots) (1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p) = 1 + \Theta_1 z + \Theta_2 z^2 + \ldots + \Theta_q z^q$$

Here $p = 2$ and $q = 4$.

$z^0$: $\psi_0 = 1$

$z^1$: $\psi_1 - \phi_1 \psi_0 = \Theta_1$

$z^2$: $\psi_2 - \phi_1 \psi_1 - \phi_2 \psi_0 = \Theta_2$

$z^3$: $\psi_3 - \phi_1 \psi_2 - \phi_2 \psi_1 = \Theta_3$

$z^4$: $\psi_4 - \phi_1 \psi_3 - \phi_2 \psi_2 = \Theta_4$

$z^5$: $\psi_5 - \phi_1 \psi_4 - \phi_2 \psi_3 = 0$

and observe that in powers larger than 4 we have the recursion: $\psi_j - \sum_{k=1}^{\infty} \phi_k \psi_{j-k} = 0$

Ok, in summary, $\psi_j = \sum_{k=1}^{p} \phi_k \psi_{j-k}, j \geq 5$

with initial conditions $\psi_0 = 1$ and $\psi_j = \sum_{k=1}^{q} \Theta_k \psi_{j-k} = \Theta_j$

for $j = 1, 2, 3, 4$.

(in the initial conditions above I used the fact that $\phi_3 = 0$ and $\phi_4 = 0$ since we have $p = 2$ here).
Invertible representation: $\pi(B)X_t = W_t$ et, equivalently,

there exists weights $\pi_j$ s.t. $\sum_{j=0}^{\infty} \pi_j X_{t-j} = W_t$, with $\pi_0 = 1$.

Since $X_t \sim ARMA(2,4) \Rightarrow \phi(B)X_t = \theta(B)W_t \Rightarrow W_t = \frac{\phi(B)}{\theta(B)}$

and thus the weights $\pi_j$ will be determined by equating the powers of $z$ in the identity: $\pi(B)\theta(B) = \phi(z)$ (->

$$(\pi_0 + \pi_1 z + \pi_2 z^2 + \cdots)(1 + \theta_1 z + \theta_2 z^2 + \theta_3 z^3 + \theta_4 z^4) = 1 - \phi_1 z - \phi_2 z^2$$

$x^2$: $\pi_0 = 1$

$x^1$: $\pi_1 + \pi_0 \theta_1 = -\phi_1$

$x^2$: $\pi_2 + \pi_1 \theta_1 + \pi_0 \theta_2 = -\phi_2$

$x^3$: $\pi_3 + \pi_2 \theta_1 + \pi_1 \theta_2 + \pi_0 \theta_3 = 0$

and for any power larger than polynomials

$z^4$: $\pi_4 + \pi_3 \theta_1 + \pi_2 \theta_2 + \pi_1 \theta_3 + \pi_0 \theta_4 = 0$

$z^5$: $\pi_5 + \pi_4 \theta_1 + \pi_3 \theta_2 + \pi_2 \theta_3 + \pi_1 \theta_4 = 0$

$z^6$: $\pi_6 + \pi_5 \theta_1 + \pi_4 \theta_2 + \pi_3 \theta_3 + \pi_2 \theta_4 = 0$

and for $j \geq 5$ we have

In summary, $\pi_j = -\frac{4}{k=1} \pi_j - k \theta_k$

from $j > 5$ we have $\pi_j = -\frac{4}{k=1} \pi_j - k \theta_k$

with initial conditions $\pi_0 = 1$ and

$\pi_j = -\sum_{k=1}^{4} (\pi_j - k \theta_k + \phi_k)$

again, I have used $\phi_3 = \phi_4 = 0$. 

$$\pi_j = -\sum_{k=0}^{4} \pi_{j-k} \theta_k$$
PROBLEM 3: Since the process is AR(2) ⇒ mean is constant (Stationarity).

a) You can find the expectation of $X_t$ in at least 2 ways.

Method 1: $E(X_t) = 3 + 0.3E(X_{t-1}) + 0.1E(X_{t-2}) + E(Z_t)$

\[ \mu = 3 + 0.3\mu + 0.1\mu \]

Solving for $\mu$ we get $\mu = 5$.

Method 2: Replace $X_t$ by $X_t - \mu$ in the expression given. ⇒ And note that when doing so, we should have no intercept.

$X_t = \mu (1-0.3-0.1) + 0.3 X_{t-1} + 0.1 X_{t-2} + Z_t$

But $X_t = 3 + 0.3 X_{t-1} + 0.1 X_{t-2} + Z_t$

⇒ $\mu (1-0.3-0.1) = 3$ ⇒ $\mu = 5$.

b) To use Yule Walker, we need to have a zero-mean process. Thus define $Y_t = X_t - 5$ and note $E(Y_t) = 0$.

Thus $Y_t = 0.3 Y_{t-1} + 0.1 Y_{t-2} + Z_t$. To get the Y-W equations, we multiply both sides by $Y_{t-h}$ and take expectations.

$⇒ E(Y_t Y_{t-h}) = 0.3 E(Y_{t-1} Y_{t-h}) + 0.1 E(Y_{t-2} Y_{t-h}) + E(Z_t Y_{t-h})$.

Recall that $Y_{t-h} - Y_{t-h} - 0.3 Y_{t-h} - 0.1 Y_{t-h} = 0$ ⇒ $s(h) - 0.3s(h-1) - 0.1s(h-2) = 0$

Thus $s(h) = 0.3 s(h-1) - 0.1 s(h-2)$

Recall that $s(0) = 0$ and $s(1) = 1$ ⇒

If $h = 1$ ⇒ $s(1) - 0.3 s(0) - 0.1 s(1) = 0$ ⇒ $s(1) = 0.3 / 0.9 = 0.33$

If $h = 2$ ⇒ $s(2) - 0.3 s(1) - 0.1 s(0) = 0$ ⇒ $s(2) = 0.3 \times 0.33 + 0.1 = 0.26$

If $h = 3$ ⇒ $s(3) - 0.3 s(2) - 0.1 s(1) = 0$ ⇒ $s(3) = 0.3 \times 0.26 + 0.1 \times 0.33 = 0.233$
3c) More precise = smaller variance since both estimators are unbiased.

\[
\text{var}(\hat{\mu}_2) = \frac{1}{4} \text{var}(X_1 + X_2) = \frac{1}{4} \left( 2\sigma^2(1) + 2\sigma^2(1) \right) = \frac{\sigma^2(1)}{4} \times \frac{2 + 2\sigma^2(1)}{4}
\]

\[
\text{var}(\hat{\mu}_3) = \frac{1}{9} \text{var}(X_1 + X_2 + X_3) = \frac{1}{9} \left( 3\sigma^2(1) + 4\sigma^2(1) + 2\sigma^2(2) \right) = \frac{\sigma^2(1)}{9} \times \frac{3 + 4\sigma^2(1) + 2\sigma^2(2)}{9}
\]

Thus, using the result in part b), we get \(\text{var}(\hat{\mu}_2) = 0.678\sigma^2(1)\) and \(\text{var}(\hat{\mu}_3) = 0.538\sigma^2(1)\).

Since \(\sigma^2(1) < \infty\) (by stationarity), we have that \(\text{var}(\hat{\mu}_3) < \text{var}(\hat{\mu}_2)\), and hence conclude \(\hat{\mu}_3\) is a more precise estimate of \(\mu\).

This shows that it is beneficial to collect more points.