\[ f(x, y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1 - \rho^2}} \exp \left( -\frac{1}{2(1 - \rho^2)} \left[ \frac{(x - \mu_x)^2}{\sigma_x^2} + \frac{(y - \mu_y)^2}{\sigma_y^2} - \frac{2\rho(x - \mu_x)(y - \mu_y)}{\sigma_x \sigma_y} \right] \right) \]

One of the earliest uses of this bivariate density was as a model for the joint distribution of the heights of fathers and sons. The density depends on five parameters:

\[-\infty < \mu_x < \infty \quad -\infty < \mu_y < \infty \]
\[\sigma_x > 0 \quad \sigma_y > 0 \]
\[-1 < \rho < 1 \]

The contour lines of the density are the lines in the \(xy\) plane on which the joint density is constant. From the equation above, we see that \(f(x, y)\) is constant if

\[
\frac{(x - \mu_x)^2}{\sigma_x^2} + \frac{(y - \mu_y)^2}{\sigma_y^2} - \frac{2\rho(x - \mu_x)(y - \mu_y)}{\sigma_x \sigma_y} = \text{constant}
\]

The locus of such points is an ellipse centered at \((\mu_x, \mu_y)\). If \(\rho = 0\), the axes of the ellipse are parallel to the \(x\) and \(y\) axes, and if \(\rho \neq 0\), they are tilted. Figure 3-7 shows several bivariate normal densities, and Figure 3-8 shows the corresponding elliptical contours.

Figure 3-7. Bivariate normal densities with \(\mu_x = \mu_y = 0\) and \(\sigma_x = \sigma_y = 1\) and
(a) \(\rho = 0\), (b) \(\rho = .3\), (c) \(\rho = .6\), (d) \(\rho = .9\).