SOLUTIONS TO THIRD IN-CLASS EXAM

1. We use \( r \) (the radius of the semicircle) as our basic variable. Then the perimeter of the window is \( \pi r + 2h + w \), where \( h \) is the height and \( w \) is the width of the rectangular part of the window. From the picture \( w = 2r \) (the diameter of the semicircle is the width of the rectangle), and then the perimeter formula says that \( \pi r + 2h + 2r = 120 \) or \( h = 60 - (\frac{\pi}{2} + 1)r \). The area is \( \frac{1}{2} \pi r^2 + hw \), so the function we want to maximize is

\[
A = \frac{1}{2} \pi r^2 + (60 - (\frac{\pi}{2} + 1)r)2r = -\left(\frac{\pi}{2} + 2\right)r^2 + 120r.
\]

Then

\[
\frac{dA}{dr} = -\left(\pi + 4\right)r + 120,
\]

which is zero when \( 120 = (\pi + 4)r \), so the dimensions are

\[
r = \frac{120}{\pi + 4}, \quad h = \frac{120}{\pi + 4}, \quad w = \frac{240}{\pi + 4}.
\]

2. We have

\[
f'(x) = (x - 2)^{1/2} + \frac{1}{2}x(x - 2)^{-1/2} = \left(\frac{3}{2}x - 2\right)(x - 2)^{-1/2},
\]

and

\[
f''(x) = \frac{3}{2}(x - 2)^{-1/2} - \frac{1}{2}(x - 2)(x - 2)^{-3/2} = \left(\frac{3}{4}x - 2\right)(x - 2)^{-3/2}.
\]

Since \( x - 2 \) must be positive, it follows that \( f'(x) > 0 \) for all \( x \), so \( f \) is increasing on \( (2, \infty) \).

In addition, \( f''(x) < 0 \) if \( 2 < x < 8/3 \) and \( f''(x) > 0 \) if \( x > 8/3 \), so \( f \) is concave down up on \( (8/3, \infty) \) and concave down on \( (2, 8/3) \).

The graph is at the end of the solutions.

3. The derivative is

\[
s'(t) = \cos t + \sin t.
\]
So the critical points are where $s' = 0$, that is, $t = \frac{3\pi}{4}$ (Since that's where $\tan t = -1$ in the interval $I$), and at the endpoints $t = 0$ and $t = \pi$. We then evaluate:

$$s(0) = -1 \quad \text{minimum}$$

$$s\left(\frac{3\pi}{4}\right) = \sqrt{2} \quad \text{maximum}$$

$$s(\pi) = 1.$$

4. Now

$$r'(s) = 3 + \frac{2}{5}s^{-3/5},$$

This is zero when $s = -(15/2)^{-5/3}$ and it's undefined when $s = 0$, so these are the critical points.

Since

$$r''(s) = -\frac{6}{25}s^{-4/5},$$

the function is concave down on the intervals $(-\infty, 0)$ and $(0, \infty)$. Because the first derivative is undefined at $s = 0$, the graph must look like a script “m”. Therefore the function has a local minimum when $s = 0$ and a local minimum when $s = -(15/2)^{-5/3}$.

It's also possible to use the first derivative test. If $s < -(15/2)^{-5/3}$, then $r'(s) > 0$, so $r$ is increasing on this range. If $-(15/2)^{-5/3} < s < 0$, then $r'(s) < 0$, so $r$ is decreasing on this range. If $s > 0$, then $r'(s) > 0$, so $r$ is increasing on this range.