1. (a) Because \( \text{rank}(AB) \leq \min \{ \text{rank}(A), \text{rank}(B) \} \) and \( PxX = X \), we have

\[
\text{rank}(X) = \text{rank}(PxX) \\
= \text{rank}(X(X'X)^{-1}X'X) \\
\leq \text{rank}(X'X) \\
\leq \text{rank}(X)
\]

\[\therefore \text{rank}(X) = \text{rank}(X'X)\]
(b) Similarly,

\[ \text{rank}(X) = \text{rank}(P_x X) \]

\[ \leq \text{rank}(P_x) = \text{rank}(X(X'X)^{-1}X') \]

\[ \leq \text{rank}(X) \]

By \( \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\} \)

\[ P_x X = X \]

\[ \therefore \text{rank}(X) = \text{rank}(P_x) \]
\( q = \text{rank}(C) \)
\[= \text{rank}(AX) \]
\[= \text{rank}(APxX) \]
\[\leq \text{rank}(APx) \]
\[= \text{rank}[(APx)'] \]
\[= \text{rank}[(APx)']'(APx) \]
\[= \text{rank}[APx(APx)'] \]
\[= \text{rank}[APxPx'A'] \]
\[= \text{rank}[APxA'] \]
\[= \text{rank}(AX(X'X)^{-1}X'A') \]
\[= \text{rank}(C(X'X)^{-1}C') \]
\[\leq \text{rank}(C) \]
\[= q. \]

\[\therefore \text{rank}(C(X'X)^{-1}C') = q. \]
2. (a) The design matrix $X$ is

$$X = \begin{bmatrix}
1 & 1 & 0 & -1 \\
1 & 1 & 0 & -1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 1 & -1 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix}$$
(b) i. Note that $\mu + s_i = E(Y_{i21})$.

By Result 3.7 (i) on Page 198, $\mu + s_i$ is estimable.

ii. Note that $\beta = (\mu + s_i) - (\mu + s_i - \beta) = E(Y_{i12}) - E(Y_{i11})$,

thus, by Result 3.7, $\beta$ is estimable.

By Part (i), $\mu + s_i$ is also estimable, thus, $\mu + s_i + 10 \beta$ is a l.c. of estimable functions. By Result 3.7 (ii), $\mu + s_i + 10 \beta$ is estimable.

iii. Note that $s_i - s_2 = (\mu + s_i) - (\mu + s_2) = E(Y_{i12}) - E(Y_{i22})$,

thus, by Result 3.7, $s_i - s_2$ is estimable.

iv. To prove that $\mu$ is non-estimable, we will use the property that $C^t \beta$ is estimable if and only if $C^t a = 0 \forall a$ for which $X a = 0$.

Note that $X a = 0 \Rightarrow a = [1 -1 -10]^t b$, where $b \in \mathbb{R}$.

Let $a = [1 -1 -10]^t$.

Then, $X a = 0$, but $C^t a = (1 0 0 0) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1 \neq 0$.

Therefore, $\mu$ is non-estimable.
(c) A full-column-rank matrix that has the same column space as $X$ in Part (a) is

\[ W = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \]
(d) Using $W$ as a design matrix, we get an equivalent model

\[ y = W \xi + \xi, \]  

where \( \xi = [\xi_1, \xi_2, \beta]' \), where \( \xi_1 = \mu + s_1 \), \( \xi_2 = \mu + s_2 \).

Then we have \( E(y_{im}) = \xi_1 - \beta = \xi' \xi, \) where \( \xi = (1 \, 0 \, -1)' \).

Note that the BLUE of \( \xi' \xi \) is \( \xi' \hat{\xi}_{ols} = \xi' (W'W)^{-1} W'y \),  
i.e., the BLUE of \( E(y_{im}) \) is \( \xi' (W'W)^{-1} W'y \).

We then computes the following:

\[ (W'W)^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{8} \end{bmatrix} \]  
\[ W'y = \begin{bmatrix} y_{1m} \\ y_{2m} \\ y_{3m} - y_{1m} \end{bmatrix}, \]

where \( y_{1m} = \frac{1}{3} \sum_{j=1}^{3} \sum_{k=1}^{3} i_{jk}, \) \( y_{2m} = \frac{1}{3} \sum_{j=1}^{3} \sum_{k=1}^{3} j_{jk}, \) \( y_{3m} = \frac{1}{3} \sum_{i=1}^{3} \sum_{k=1}^{3} \) j_{ik},

and \( y_{1i} = \frac{1}{3} \sum_{j=1}^{3} \sum_{k=1}^{3} i_{ik}, \)

\[ \hat{\xi}_{ols} = (W'W)^{-1} W'y = \begin{bmatrix} \bar{y}_{1m} \\ \bar{y}_{2m} \\ \frac{1}{\xi} (\bar{y}_{3m} - \bar{y}_{1m}) \end{bmatrix}, \]  
where \( \bar{y}_{im} = \frac{1}{6} y_{im} \) for \( i = 1, 2, \)

\[ \bar{y}_{jm} = \frac{1}{\xi} y_{jm} \]  
for \( j = 1, 2, 3, \)

Therefore, the BLUE of \( E(y_{im}) \) is \( (1 \, 0 \, -1)' \hat{\xi}_{ols} \)

\[ = \bar{y}_{im} - \frac{1}{\xi} (\bar{y}_{3m} - \bar{y}_{1m}). \]
(e) i. \( \hat{\mu} + \hat{s}_1 = \bar{z}_l = \bar{y}_{1..} \)

ii. \( \hat{\mu} + \hat{s}_1 + 10\hat{\beta} = \bar{z}_i + 10\hat{\beta} = \bar{y}_{1..} + 5(\bar{y}_{3..} - \bar{y}_{1..}) \)

iii. \( \hat{\beta} - \bar{y}_{2..} = \bar{y}_{1..} - \bar{y}_{2..} \)