THE AITKEN MODEL

\[ y = X\beta + \epsilon, \; \epsilon \sim (0, \sigma^2 V) \]

- Identical to the Gauss-Markov linear model except that \( \text{Var}(\epsilon) = \sigma^2 V \) instead of \( \sigma^2 I \).

- \( V \) is assumed to be a known nonsingular variance matrix.

- \( \sigma^2 \) is an unknown positive variance parameter.
A Transformation of the Model

By the Spectral Decomposition Theorem, there exists a nonsingular symmetric matrix $V^{1/2}$ such that $V^{1/2}V^{1/2} = V$.

Using $V^{-1/2}$ to denote $(V^{1/2})^{-1}$, we have

$$V^{-1/2}y = V^{-1/2}X\beta + V^{-1/2}\epsilon.$$

With $z = V^{-1/2}y$, $W = V^{-1/2}X$, and $\delta = V^{-1/2}\epsilon$, we have

$$z = W\beta + \delta, \quad \delta \sim (0, \sigma^2I)$$

because

$$\text{Var}(\delta) = \text{Var}(V^{-1/2}\epsilon) = V^{-1/2}\sigma^2VV^{-1/2}$$

$$= \sigma^2V^{-1/2}V^{1/2}V^{1/2}V^{-1/2} = \sigma^2I.$$
Thus, after transformation, we are back to the Gauss-Markov model we are familiar with.

We can apply all the results we have established previously to the Gauss-Markov model

\[ z = W\beta + \delta, \quad \delta \sim (0, \sigma^2 I). \]
Estimation of $E(y)$ under the Aitken Model

- Note that
  \[ E(y) = E(V^{1/2}V^{-1/2}y) = V^{1/2}E(V^{-1/2}y) = V^{1/2}E(z). \]

- Because the Gauss-Markov model holds for $z$, we already know that the best estimate of $E(z)$ is
  \[ \hat{z} = P_wz = W(W'W)^{-1}W'z \]
  \[ = V^{-1/2}X((V^{-1/2}X)'V^{-1/2}X)-(V^{-1/2}X)'V^{1/2}y \]
  \[ = V^{-1/2}X(X'V^{-1/2}V^{-1/2}X)-X'V^{-1/2}V^{-1/2}y \]
  \[ = V^{-1/2}X(X'V^{-1}X)-X'V^{-1}y. \]

- Thus, to estimate $E(y) = V^{1/2}E(z)$, we should use
  \[ V^{1/2}\hat{z} = V^{1/2}V^{-1/2}X(X'V^{-1}X)-X'V^{-1}y = X(X'V^{-1}X)-X'V^{-1}y. \]
Likewise, if $C\beta$ is estimable, we know the BLUE is the ordinary least squares (OLS) estimator.

$$C(W'W)^{-1}W'z = C(X'V^{-1/2}V^{-1/2}X)^{-1}X'V^{-1/2}V^{-1/2}y$$

$$= C(X'V^{-1}X)^{-1}X'V^{-1}y.$$ 

$C(X'V^{-1}X)^{-1}X'V^{-1}y = C\hat{\beta}_V$ is called a Generalized Least Squares (GLS) estimator.
\[ \hat{\beta}_V = (X'V^{-1}X)^{-1}X'V^{-1}y \text{ is a solution to the Aitken Equations:} \]

\[ X'V^{-1}Xb = X'V^{-1}y \]

which follow from the Normal Equations

\[ W'Wb = W'z \iff X'V^{-1/2}V^{-1/2}Xb = X'V^{-1/2}V^{-1/2}y \]

\[ \iff X'V^{-1}Xb = X'V^{-1}y. \]
Recall that solving the Normal Equations is equivalent to minimizing

$$(z - Wb)'(z - Wb)$$

over $b \in IR^p$.

Note that

$$(z - Wb)'(z - Wb) = (V^{-1/2}y - V^{-1/2}Xb)'(V^{-1/2}y - V^{-1/2}Xb)$$

$$= \|V^{-1/2}y - V^{-1/2}Xb\|^2$$

$$= \|V^{-1/2}(y - Xb)\|^2$$

$$= (y - Xb)'V^{-1}(y - Xb).$$
Thus, $\hat{\beta}_V = (X'V^{-1}X)^{-1}X'V^{-1}y$ is a solution to this generalized least squares problem.

When $V$ is diagonal, the term “Weighted Least Squares” (WLS) is often used instead of GLS.
An unbiased estimator of $\sigma^2$ is

\[
\frac{z'(I - P_W)z}{n - \text{rank}(W)} = \frac{\| (I - P_W)z \|^2}{n - \text{rank}(W)} = \frac{\| (I - W(W'W)'W')z \|^2}{n - \text{rank}(W)} = \frac{\| (I - V^{-1/2}X(X'V^{-1}X) - X'V^{-1}/2)V^{-1/2}y \|^2}{n - \text{rank}(V^{-1/2}X)}
\]
\[
\begin{align*}
&= \left\| V^{-1/2}y - V^{-1/2}X(X'V^{-1}X)^{-1}X'V^{-1}y \right\|^2 \\
&= \frac{\left\| V^{-1/2}(y - X(X'V^{-1}X)^{-1}X'V^{-1}y) \right\|^2}{n - \text{rank}(X)} \\
&= \frac{\left\| V^{-1/2}(y - X\hat{\beta}_V) \right\|^2}{n - r} \\
&= \frac{(y - X\hat{\beta}_V)'V^{-1}(y - X\hat{\beta}_V)}{n - r} \\
&= \hat{\sigma}_V^2.
\end{align*}
\]
Inference Under the Normal Theory Aitken Model

- The Normal Theory Aiken Model:
  \[ y = X\beta + \epsilon, \quad \epsilon \sim N(0, \sigma^2 V). \]

- Under the Normal Theory Aitken Model, we can back transform to convert known formulas in terms of \( z \) and \( W \) to formulas in terms of \( y \) and \( X \) to allow inference about estimable \( C\beta \) under the Normal Theory Aitken Model.