Equivalence of the Reduced versus Full Model F test and the F test of $C \beta = d$

<table>
<thead>
<tr>
<th>Storage Temperature</th>
<th>20°C</th>
<th>30°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Storage Time</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 months</td>
<td>2 5</td>
<td>9 12 15</td>
</tr>
<tr>
<td>6 months</td>
<td>6 6 7 7</td>
<td>16</td>
</tr>
</tbody>
</table>

```r
time = factor(rep(c(3, 6), each = 5))
temp = factor(rep(c(20, 30, 20, 30), c(2, 3, 4, 1)))
y = c(2, 5, 9, 12, 15, 6, 6, 7, 7, 16)
d = data.frame(time, temp, y)
```
d
time  temp  y
1     3     20   2
2     3     20   5
3     3     30   9
4     3     30  12
5     3     30  15
6     6     20   6
7     6     20   6
8     6     20   7
9     6     20   7
10    6     30  16

full=lm(y~time+temp+time:temp,data=d)
```r
model.matrix(full)

<table>
<thead>
<tr>
<th></th>
<th>Intercept</th>
<th>time6</th>
<th>temp30</th>
<th>time6:temp30</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
```
coef(full)

(Intercept)      time6   temp30  time6:temp30
  3.5     3.0      8.5       1.0

# temp 20      temp 30
# -------------------------
# time 3       mu      mu+temp30
# time 6      mu+time6   mu+time6+temp30+time6:temp30
```R
test=function(lmout,C,d=0){
  b=coef(lmout)
  V=vcov(lmout)
  dfn=nrow(C)
  dfd=lmout$df
  Cb.d=C%*%b-d
  Fstat=drop(t(Cb.d)%*%solve(C%*%V%*%t(C))%*%Cb.d/dfn)
  pvalue=1-pf(Fstat,dfn,dfd)
  list(Fstat=Fstat,pvalue=pvalue)
}

Coverall=matrix(c( 0,1,0,0,
                  0,0,1,0,
                  0,0,0,1
                  0,0,0,1
                ),nrow=3,byrow=T)
```
test(full,Coverall)

$Fstat$

[1] 13.53191

$pvalue$

[1] 0.004438826

reduced = lm(y ~ 1, data=d)

model.matrix(reduced)

(Intercept)

    1 1 2 1 3 1 4 1 5 1 6 1 7 1 8 1 9 1 10 1
function rvsf(reduced, full) {
  sser = deviance(reduced)
  ssef = deviance(full)
  dfer = reduced$df
  dfef = full$df
  dfn = dfer - dfef
  Fstat = (sser - ssef) / dfn / (ssef / dfef)
  pvalue = 1 - pf(Fstat, dfn, dfef)
  list(Fstat = Fstat, dfn = dfn, dfd = dfef, pvalue = pvalue)
}

rvsf(reduced, full)
$Fstat
[1] 13.53191

$dfn
[1] 3
```r
$dfd
[1] 6

$pvalue
[1] 0.004438826

anova(reduced, full)
Analysis of Variance Table

Model 1: y ~ 1
Model 2: y ~ time temp time:temp

<table>
<thead>
<tr>
<th>Res.Df</th>
<th>RSS</th>
<th>Df</th>
<th>Sum of Sq</th>
<th>F</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>182.5</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>23.5</td>
<td>6</td>
<td>159</td>
<td>13.532</td>
<td>0.004439 **</td>
</tr>
</tbody>
</table>

---
Signif. codes: 0 ***, 0.001 **, 0.01 *, 0.05 ., 0.1 , 1
```
Cinteraction=matrix(c(0,0,0,1),nrow=1,byrow=T)

test(full,Cinteraction)
$Fstat
[1] 0.1225532

$pvalue
[1] 0.7382431

anova(full)
Analysis of Variance Table
Response: y

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td>1</td>
<td>0.10</td>
<td>0.100</td>
<td>0.0255</td>
<td>0.878292</td>
</tr>
<tr>
<td>temp</td>
<td>1</td>
<td>158.42</td>
<td>158.420</td>
<td>40.4477</td>
<td>0.000709 ***</td>
</tr>
<tr>
<td>time:temp</td>
<td>1</td>
<td>0.48</td>
<td>0.480</td>
<td>0.1226</td>
<td>0.738243</td>
</tr>
<tr>
<td>Residuals</td>
<td>6</td>
<td>23.50</td>
<td>3.917</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
summary(full)

Call:
lm(formula = y ~ time temp time:temp, data = d)

Residuals:
       Min        1Q  Median        3Q        Max
-3.000e+00 -5.000e-01 -7.606e-17  5.000e-01  3.000e+00

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)     3.500      1.399   2.501  0.04646 *
time6           3.000      1.714   1.750  0.13062
temp30          8.500      1.807   4.705  0.00331 **
time6:temp30    1.000      2.857   0.350  0.73824

Residual standard error: 1.979 on 6 degrees of freedom
Multiple R-squared: 0.8712, Adjusted R-squared: 0.8068
F-statistic: 13.53 on 3 and 6 DF, p-value: 0.004439
```r
additive = lm(y ~ time + temp, data = d)
model.matrix(additive)

<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>1</td>
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<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>0</td>
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<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
```
coef(additive)
(Intercept)       time6      temp30
  3.26        3.36        8.90

#          temp  20   temp  30
#          -------------------------
# time 3   mu        mu+temp30
#
# time 6   mu+time6  mu+time6+temp30
#

rvsf(additive,full)
$Fstat
[1] 0.1225532

$dfn
[1] 1
$\text{dfd}$
[1] 6

$pvalue$
[1] 0.7382431

\textbf{anova(additive,full)}

Analysis of Variance Table

Model 1: y ~ time temp
Model 2: y ~ time temp time:temp

<table>
<thead>
<tr>
<th>Res.Df</th>
<th>RSS</th>
<th>Df</th>
<th>Sum of Sq</th>
<th>F</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>23.98</td>
<td>7</td>
<td>0.48</td>
<td>0.1226</td>
<td>0.7382</td>
</tr>
<tr>
<td>2</td>
<td>23.50</td>
<td>6</td>
<td>1</td>
<td>0.48</td>
<td>0.1226</td>
</tr>
</tbody>
</table>
drop1(full, test="F")

Single term deletions

Model:
\[ y \sim \text{time} + \text{temp} + \text{time:temp} \]

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>RSS</th>
<th>AIC</th>
<th>F value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;none&gt;</td>
<td>23.50</td>
<td>16.544</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>time:temp</td>
<td>1</td>
<td>0.48</td>
<td>23.98</td>
<td>14.746</td>
<td>0.1226</td>
</tr>
</tbody>
</table>

anova(additive)

Analysis of Variance Table

Response: y

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td>1</td>
<td>0.10</td>
<td>0.100</td>
<td>0.0292</td>
</tr>
<tr>
<td>temp</td>
<td>1</td>
<td>158.42</td>
<td>158.420</td>
<td>46.2444</td>
</tr>
<tr>
<td>Residuals</td>
<td>7</td>
<td>23.98</td>
<td>3.426</td>
<td></td>
</tr>
</tbody>
</table>

---

Signif. codes: 0 `***' 0.001 `**' 0.01 `*' 0.05 `.' 0.1 ` ' 1
drop1(additive,test="F")

Single term deletions

Model:

\(y \sim time + temp\)

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>RSS</th>
<th>AIC</th>
<th>F value</th>
<th>Pr((F))</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;none&gt;</td>
<td>23.98</td>
<td>14.746</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>time</td>
<td>1</td>
<td>23.52</td>
<td>47.50</td>
<td>19.581</td>
<td>0.0343966  *</td>
</tr>
<tr>
<td>temp</td>
<td>1</td>
<td>158.42</td>
<td>182.40</td>
<td>33.036</td>
<td>0.0002531 ***</td>
</tr>
</tbody>
</table>

---

Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

anova(lm(y~temp+time,data=d))

Analysis of Variance Table

Response: y

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr((&gt;F))</th>
</tr>
</thead>
<tbody>
<tr>
<td>temp</td>
<td>1</td>
<td>135.00</td>
<td>135.000</td>
<td>39.4078</td>
</tr>
<tr>
<td>time</td>
<td>1</td>
<td>23.52</td>
<td>23.520</td>
<td>6.8657</td>
</tr>
<tr>
<td>Residuals</td>
<td>7</td>
<td>23.98</td>
<td>3.426</td>
<td></td>
</tr>
</tbody>
</table>
Previously we saw how to test for time main effects and temp main effects in the full model by testing \( H_0: C \beta = d \).

It is possible but not as easy to test for these main effects using the reduced versus full model approach. We will use the test for time main effects as an example.

We need to find a matrix whose column space allows for temp main effects and time-by-temp interaction but no time main effects.

It is natural to try the following model specification.
o = lm(y ~ temp + time:temp, data = d)
model.matrix(o)

<table>
<thead>
<tr>
<th></th>
<th>(Intercept)</th>
<th>temp30</th>
<th>temp20:time6</th>
<th>temp30:time6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
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<td>0</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Examination of this design matrix shows that the cell means are modeled as

```
# temp 20          temp 30
# --------------------------
# time 3   mu               mu + temp30
# time 6   mu + temp20:time6  mu + temp30 + temp30:time6
```
#It is easy to see that this is just the full model
#in which each treatment group is allowed to have
#its own mean. Thus, we can't use this code to
#obtain our reduced model fit.

#One way to obtain the test for time main effects
#by comparing a reduced and full model is as follows.

full=lm(y~time+temp+time:temp,data=d)

C=matrix(c(
+ 0,1,0,.5
+ ),nrow=1,byrow=T)

B=matrix(c(
+ 1,0,0,0,
+ 0,0,1,0,
+ 0,0,0,1,
+ 0,1,0,.5
+ ),nrow=4,byrow=T)
#Note that $X \beta = X B^{-1} B \beta$.
#
#Let $W = X B^{-1}$ and $\alpha = B \beta$.
#
#Then $C \beta = 0$ is equivalent to $\alpha_4 = 0$. 
```r
W = model.matrix(full) %*% solve(B)

W0 = W[, 1:3]

newfull = lm(y ~ W - 1)

newreduced = lm(y ~ W0 - 1)

anova(newreduced, newfull)

Analysis of Variance Table

Model 1: y ~ W0 - 1
Model 2: y ~ W - 1

<table>
<thead>
<tr>
<th>Res.Df</th>
<th>RSS</th>
<th>Df</th>
<th>Sum of Sq</th>
<th>F</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>47.02</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>23.50</td>
<td>1</td>
<td>23.52</td>
<td>6.0051</td>
<td>0.04975*</td>
</tr>
</tbody>
</table>
```
rvsf(newreduced,newfull)
$F_{stat}$
[1] 6.005106

$df_{n}$
[1] 1

$df_{d}$
[1] 6

$p_{value}$
[1] 0.04975481

test(full,C)
$F_{stat}$
[1] 6.005106

$p_{value}$
[1] 0.04975481