Inference for \( C^2 \) under the Gaussian Normal Theory and Markov Models
Throughout this set of notes, we assume the normal theory Gauss-Markov model holds.

\[ y = X\beta + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I). \]
We have previously established

\[ \hat{c}_β \sim N(c_β, σ^2 c(x'x)^{-1}c') \]

for estimable \( c_β \) and

\[ \frac{(n-r)\hat{σ}^2}{σ^2} \sim X^2_{n-r}. \]
Suppose we wish to test a null hypothesis of the form

\[ H_0: \beta = \theta \]

against

\[ H_A: \beta \neq \theta \].
$H_0: C_{\beta} = d \text{ is testable}$

if $\text{rank } (C_{\beta})_{q\times p} = q$,

$C_{\beta}$ is estimable,

and $d$ is a known $q\times 1$ vector for some $q > 0$. 
What test statistic shall we use to test $H_0$?

In introductory statistics, we learn to test

$H_0: \theta = \theta_0$ vs. $H_A: \theta \neq \theta_0$

using a test statistic.
of the form

\[
\frac{\hat{\theta} - \theta}{SE(\hat{\theta})}
\]

Where \(\hat{\theta}\) is an estimate of \(\theta\) and \(SE(\hat{\theta})\) is the standard error of \(\hat{\theta}\), i.e., \(SE(\hat{\theta}) = \sqrt{\text{Var}(\hat{\theta})}\).
We reject $H_0: \theta = d$ in favor of $H_a: \theta \neq d$ if and only if

$$
\left(\frac{\hat{\theta} - d}{SE(\hat{\theta})}\right)^2 = \left(\frac{\hat{\theta} - d}{\sqrt{\text{Var}(\hat{\theta})}}\right)^2 = \frac{(\hat{\theta} - d)^2}{\text{Var}(\hat{\theta})}
$$

$$
= (\hat{\theta} - d) \left[ \text{Var}(\hat{\theta}) \right]^{-1} (\hat{\theta} - d)
$$

$$
= (\hat{\theta} - d) \left[ \text{Var}(\hat{\theta}) \right]^{-1} (\hat{\theta} - d) / 1
$$

is large enough.
Thus, to test $H_0: C\hat{\beta} = d$ vs. $H_A: C\hat{\beta} = d$, it may be reasonable to consider the following test statistic:

$$(C\hat{\beta} - d)' [\text{Var} (C\hat{\beta})]^{-1} (C\hat{\beta} - d) / q$$

$$= (C\hat{\beta} - d)' [\hat{\sigma}^2 C (X'X)^{-1} C']^{-1} (C\hat{\beta} - d) / q$$
\[
\begin{align*}
\hat{\sigma}^2 & = (c_{\hat{\beta} - 2d})' [c (x'x)^{-1} c]' (c_{\hat{\beta} - 2d})/q \\
& \quad \cdot \left(\frac{\hat{\sigma}^2}{\hat{\sigma}/\hat{\sigma}^2}\right)
\end{align*}
\]
The distribution of the denominator \( \frac{s^2}{\sigma^2} \) is that of a central chi-square divided by its degrees of freedom \( n-r \), where \( r = \text{rank}(x) \).
This follows because we have previously shown

\[
\frac{(n-r)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-r}, \text{ which implies } \frac{\hat{\sigma}^2}{\sigma^2} \sim \frac{\chi^2_{n-r}}{n-r}.
\]
What about the distribution of the numerator

\[(c\hat{\beta} - d)' [\text{Var}(c\hat{\beta})]^{-1} (c\hat{\beta} - d) / q?\]
Note that $\hat{C}^\beta - d \sim N(C^\beta - d, \text{Var}(C^\beta))$

Also note that

$$\begin{bmatrix} \text{Var}(C^\beta) \end{bmatrix}^{-1} \begin{bmatrix} \text{Var}(C^\beta) \end{bmatrix} = I,$$

which is idempotent.
Thus, our previous result about the distribution of quadratic forms implies

\[(c\hat{\beta} - d)' \left[ \text{Var}(c\hat{\beta}) \right]^{-1} (c\hat{\beta} - d) \]

\[\sim \chi^2_q \left\{ (c\beta - d)' \left[ \text{Var}(c\beta) \right]^{-1} (c\beta - d) \right\} \]

Where \( q = \text{rank}(c) = \text{rank}(\left[ \text{Var}(c\beta) \right]^{-1}) \)
Thus, the numerator of the proposed test statistic also has a chi-square distribution divided by its degrees of freedom.
Because

\[ \text{Var}(c \hat{\beta}) = \sigma^2 C (x'x)^{-1} C', \]

we can rewrite the NCP as

\[ (c \beta - d)' [C(x'x)^{-1}C']^{-1} (c \beta - d) \]

\[ \equiv \delta^2 \]

\[ \sigma^2 \]
Note that if $H_0: C\beta = d$ is true, $\delta^2 = 0$ so that the quadratic form in the numerator becomes central rather than noncentral chi-square.
F Distributions

If \( U_1 \sim \chi^2_{n_1}(\theta) \) independent of \( U_2 \sim \chi^2_{n_2} \),
then \( F = \frac{U_1/n_1}{U_2/n_2} \) is said to have a noncentral \( F \) distribution with \( n_1 \) and \( n_2 \) degrees of freedom and non-centrality parameter \( \theta \). If \( \theta = 0 \), \( F \) is said to have a central \( F \) distribution.

We write \( F \sim F_{n_1, n_2}(\theta^2) \).
Suppose \( \mathbf{w} \sim \mathcal{N}(\mathbf{m}, \Sigma) \) is \( n \times 1 \). Suppose \( \mathbf{A} \) is an \( m \times n \) matrix and \( \mathbf{B} \) is a symmetric \( n \times n \) matrix satisfying \( \mathbf{A} \Sigma \mathbf{B} = 0 \). Then \( \mathbf{A} \mathbf{w} \) and \( \mathbf{w}' \mathbf{B} \mathbf{w} \) are independent.

**Proof:**

\[
\begin{bmatrix}
\mathbf{A} \\
\mathbf{B}
\end{bmatrix} \sim \mathcal{N}
\begin{bmatrix}
\mathbf{A}\\
\mathbf{B}
\end{bmatrix} \mathbf{w} \sim \mathcal{N}
\begin{bmatrix}
\mathbf{A} \\
\mathbf{B}
\end{bmatrix} \mathbf{m},
\begin{bmatrix}
\mathbf{A}\\
\mathbf{B}
\end{bmatrix} \Sigma \begin{bmatrix}
\mathbf{A}\\
\mathbf{B}
\end{bmatrix}'.
\]

\[
\begin{bmatrix}
\mathbf{A} \\
\mathbf{B}
\end{bmatrix} \Sigma \begin{bmatrix}
\mathbf{A}\\
\mathbf{B}
\end{bmatrix}' = \begin{bmatrix}
\mathbf{A}\\
\mathbf{B}
\end{bmatrix} \Sigma \begin{bmatrix}
\mathbf{A}' & \mathbf{B}'
\end{bmatrix} = \begin{bmatrix}
\mathbf{A} \Sigma \mathbf{A}' & \mathbf{A} \Sigma \mathbf{B}' \\
\mathbf{B} \Sigma \mathbf{A}' & \mathbf{B} \Sigma \mathbf{B}'
\end{bmatrix}
\]

Because \( \mathbf{B} \) is symmetric, \( \mathbf{A} \Sigma \mathbf{B}' = \mathbf{A} \Sigma \mathbf{B} = 0 \).
Thus, \( \text{Cov}(A\mathbf{w}, B\mathbf{w}) = A \Sigma B' = A \Sigma B = 0 \).

For multivariate normal random vectors that are jointly multivariate normal, zero covariance implies independence.

Thus, \( A\mathbf{w} \) and \( B\mathbf{w} \) are independent.

This implies \( A\mathbf{w} \) and \( f(B\mathbf{w}) \) are independent for any function \( f \).

Hence, \( A\mathbf{w} \) and \( \mathbf{w}'B\mathbf{w} = \mathbf{w}'BB^{-1}B\mathbf{w} = \mathbf{w}'B'BB^{-1}B\mathbf{w} = (B\mathbf{w})'B^{-1}(B\mathbf{w}) \equiv f(B\mathbf{w}) \)

are independent.
Now consider

\[ C_{\beta} = C(x'x)^{-1}x' \] and \[ SSE = y'(I-P_x)y. \]

These are independent because

\[
C(x'x)^{-1}x'(\sigma^2 I)(I-P_x) = \sigma^2 C(x'x)^{-1}(x'-x'P_x)
\]

\[
= \sigma^2 C(x'x)^{-1}(x'-x')
\]

\[ = 0. \]

Thus, \( (C_{\beta} - \delta)' \left[ C(x'x)^{-1}C' \right]^{-1} (C_{\beta} - \delta) / \sigma^2 \) and

\[ \frac{SSE}{\sigma^2} = y'(I-P_x)y / n-r \] are independent.
From our previous results, the ratio of these quadratic forms

\[
F = \frac{(c^2 - d)' [c(x'x) - c']^{-1} (c^2 - d) / 2}{\delta^2}
\]

has a non-central F distribution with non-centrality parameter

\[
(c^2 - d)' [c(x'x) - c']^{-1} (c^2 - d) / 2
\]

and degrees of freedom \( g \) and \( n-r \).
Thus, to test $H_0: C^2 = \sigma$, we compute $F$ and compare the value of $F$ to an $F_{g, n-r}$ distribution. If $F$ is so large that it seems unlikely to have been a draw from the $F_{g, n-r}$ distribution, we reject $H_0$. 
The p-value for the test of $H_0: \sigma^2 = \sigma_0$ is the probability that a central $F_{q, n-r}$ random variable would match or exceed the observed value of $F$. 