Integration over a spherical volume:

The volume element is

\[ d^3V = r^2 \sin \theta \, d\theta \, d\phi \]

with limits of integration (0 to \( r \)) for radius, (0 to \( \pi \)) for \( \theta \), and (0 to \( 2\pi \)) for \( \phi \). For axial symmetry, i.e. quantities independent of \( \phi \), we have

\[ d^3V = 2\pi r^2 \sin \theta \, d\theta \, dr \]

so that the integral becomes

\[ 2\pi \int_0^R \left[ \int_0^\pi Q(r, \theta) \sin \theta \, d\theta \right] r^2 \, dr \]

This means: Do the integral over \( \theta \) and then integrate the result (which will be a function of \( r \) only) over \( r \).

**I. Mean density.** This is just the total mass divided by the total volume. Total mass is the integral of the density, \( \rho \), over the volume; total volume is the integral with \( Q = 1 \). Both of these are independent of \( \theta \) as well as \( \phi \), so the integral simplifies:

\[ M = \int_0^R \rho(r) \, 4\pi r^2 \, dr \]

so

\[ \rho = \frac{(4\pi/V)}{\int_0^R \rho(r) \, r^2 \, dr} \text{ with the volume } V = (4\pi) \int_0^R r^2 \, dr \]
Revised Notes for Week 4.

for a spherically symmetric body. (Confirm that if the density is constant and = \( \rho \) everywhere inside \( R \), this gives mean density = \( \rho \)).

In the box on page 72, formulae are given for a planet with two discrete densities - a core density and a mantle density. One can recover the discrete case from the more general formulation by separating the integral into a series of constant density shells - one density out to \( r_1 \), another one from \( r_1 \) to \( r_2 \), etc; for core + mantle this is:

\[
\rho = (\frac{4\pi}{M}) \int_0^{r_1} \rho_1 r^2 \, dr + (\frac{4\pi}{M}) \int_{r_1}^{R} \rho_2 r^2 \, dr
\]

2. Rotation.

The angular velocity of a uniformly rotating object is \( \omega = (2\pi/P) \) where \( P \) is the period of rotation. For a uniformly rotating (solid) body \( \omega \) is the same everywhere inside.

The angular momentum of an infinitesimal mass \( dm \) about an axis is

\[
L = dm \mathbf{v} \times \mathbf{R}
\]

where \( \mathbf{R} \) is the perpendicular vector to the axis of rotation. \( dm \) is just \( \rho \, d^3 \text{Vol} \).

For a uniformly rotating (solid) body, \( L = I \omega \).

Applying this to a uniformly rotating spherical body we find:

\[
\mathbf{R} = r \sin \theta
\]

\[
\mathbf{v} \times \mathbf{R} = \mathbf{v}(r \sin \theta) \sin \theta
\]

So:

\[
I = L/\omega = \int (r \mathbf{v} \sin^2 \theta \, dm /\omega) = \int r^2 \sin^2 \theta \, dm
\]

For a ring of constant \( \theta \) and \( r \),

\[
dm = \rho \, d^3V = \rho \, 2\pi r^2 \sin \theta \, dr \, d\theta,
\]
so for a uniformly spherical distribution of mass

\[ I = [2\pi \int_0^{\pi} \sin^3 \theta \, d\theta] \left[ \int_0^R \rho(r) \, r^4 \, dr \right] = (8/3)\pi \int_0^R \rho(r) \, r^4 \, dr \]

(Note this trick: separation of the integral over two variables into two simple integrals. Works whenever \( Q(r, q) = Q_1(r) \, Q_2(q) \).)

For uniform density this = \( (8/15) \pi \rho \, R^5 \), and the result may also be expressed as \( I = 2/5 \, M \, R^2 \).

**Gravitational potential energy:**

The net gravitational potential energy of a sphere of radius \( R \), mass \( M \) is

\[ \text{GPE} = -\int_0^R (GM_r \rho/ r) \, 4\pi r^2 \, dr \]

where \( M_r \) is the mass of material interior to \( r \),

\[ M_r = \int_0^r 4\pi r^2 \, \rho \, dr \]

To put this inside the integral for GPE we first change \( r \) to a dummy variable \( x \):

\[ \text{GPE} = -\int_0^R (G\int_0^r 4\pi x^2 \, \rho \, dx/ r) \, 4\pi r^2 \, dr \]

(What this means is: Do the integral in the [] and then plug the resulting function of \( r \) into the main integral. Note that it is important to pay attention to the limits of integration - otherwise one doesn't get the right function of \( r \)!) In the case of uniform density, \( M_r \) simplifies to \((4/3 \pi r^3 \rho)\) and we get a manageable equation for the GPE:

\[ \text{GPE} = -\int_0^R G(4/3 \pi r^3 \rho) \, \rho/ r) \, 4\pi r^2 \, dr \]

\[ = - (16\pi^2/3) \, G \, \rho^2 \int_0^R r^4 \, dr \]

and we can find \( \text{GPE} = - (\text{numerical factor}) \, GM^2/R \).

For a spherical body, \( \text{GPE} = \text{constant on a spherical surface}; \) moving material from inside that surface to outside it increases the potential energy, so the least-energy state is a sphere.
An interesting case:

Suppose that a mass m is moved a small radial distance, \( \Delta r \), at about the radius of the planet, R. What is the change in GPE?

\[
\Delta \text{GPE} = \text{GPE}_{\text{after}} - \text{GPE}_{\text{before}} = \left( \frac{-GMm}{r+\Delta r} \right) - \left( \frac{-GMm}{r} \right)
\]

The radius of the Earth is about 6400 km. Suppose you raise a mass of 1 kg to a height of 1 m above the surface of the Earth - what does this equation give? If you just plug in the numbers, your calculator will give you a nonsense result. Try it!

There is a very important trick to use when faced with a small difference between two large terms - as above. It is based on the properties of numbers of the form 1+x with \( x \ll 1 \):

\[
\Delta \text{GPE} = \left( \frac{-GMm}{r+\Delta r} \right) - \left( \frac{-GMm}{r} \right) = \left[ \frac{GMm}{r} \right] \left[ 1 - \frac{r}{r+\Delta r} \right]
\]

Expressions of the form \((1+x)^n\) with \( x \ll 1 \) may be expanded in series as

\[
(1+x)^n = 1 + nx + ...
\]

where, for \( x \) small enough, only the first term needs to be kept. So

\[
[1 - \frac{1}{1+x}] = [1 - (1+x)^{-1}] \approx [1-(1-x)] \approx x
\]

and

\[
\Delta \text{GPE} \approx \left( \frac{GMm}{r^2} \right) \Delta r = mg \Delta r,
\]

defining the surface gravity as \( g = GM/r^2 \).

This is sometimes written "\( \text{GPE} = mgh \)" in elementary physics classes. Under what conditions can you safely (accurately) use the \( mgh \) formula?
Departure from spherical symmetry:

What we just showed was that for a non-rotating body the spherical surface, surface of constant \( r \), is an equipotential surface, and raising material through an equipotential surface requires doing work (against \( g \! \) !)

In a non-rotating body, the equipotential surfaces are spherical. Since the lowest energy state is one where the body has the shape of the equipotential surface, a rotating body will generally not be exactly spherical. In a system rotating with the planet, the effective or apparent gravity is

\[
g_{\text{eff}} = (g - \omega^2 r \sin \text{ (latitude)}) \text{ or } g_{\text{eff}} = (g - \omega^2 r \sin \theta) \text{ if } \theta \text{ is the polar angle,}
\]

and the equipotential surfaces become oblate spheroids (slightly pancake-flattened). The lower effective gravity at the equator compresses the rock less there than at the poles.

What happens when \( \omega = \sqrt{(g/r)} \)??

Mountains and buoyancy:

Consider a block of material of density \( \rho_1 \) floating in material of density \( \rho_2 \). It displaces a volume \( V_2 \) of the fluid. The equilibrium position is such that the mass of the fluid displaced by the floating body equals the mass of the floating body:

\[
V_2 \rho_2 = V_1 \rho_1.
\]

On a purely solid planet, a mountain can be built up until its weight overcomes the material strength of the material supporting it. (See Chapters 5 and 6 for more about this aspect.) On a molten planet with floating continents, mountains are like the tip of the iceberg floating in the ocean - they require a large mass of less dense material below them to keep them up. So both the composition - density - of the solid crust relative to the molten interior and the strength of the material are important factors limiting mountain heights.
Why are moons and planets mostly round?

Bodies in the solar system that are larger than 1000 km generally quite round and relatively smooth while bodies that are tens of km across are often very oblong or block-shaped, even with one dimension more than twice another. Why? Here are some pieces of the puzzle:

(a) Generally only large solar system bodies have fluid interiors; smaller bodies are solid throughout.

(b) The pressure inside a planet = the weight of the rock (etc.) above you.

Another way to look at this: A giant spaceship (many km across, mean density around 0.5) is to be built to carry a few thousand colonists on a slow (multi-century!) trip to a nearby star system. Which of these designs would you recommend, and why?

Non-rotating sphere

Rotating spheroid

Rotating cylinder