BLADE ELEMENT THEORY

Assumptions:
- The blade is composed of aerodynamically independent, narrow strips or elements.
- A differential blade element of chord C and width dr, located at a radius r from the rotor axis is considered as an airfoil section.

Theory:
The airscrew is advancing at a speed of V and the velocity at the disc is given by $V_d = V_0(1 + a) = V_0 + v$.
The slipstream behind an airscrew rotates in the same sense as the blades about the airscrew axis (z-axis).
The angular velocity of the airscrew blades are $\Omega$ and the angular velocity of the flow in the plane of the blades is $a_1\Omega$ where $a_1$ is a constant for the element considered.
This element has a linear velocity in the plane of the rotation of $\Omega r$ and the flow is itself rotating in the same plane and sense with an angular velocity $a_1\Omega$.
Thus the relative linear velocity of the element relative to the air in this plane is $\Omega r(1 - a_1)\omega r (\Omega - \omega)$.
$\theta$ is the geometric helix angle of the element measured between the zero lift-line of the element and the rotor disc.
$\alpha$ is the angle between the relative velocity $V_R$ and the chord.
$\phi$ is the angle between the resultant velocity $V_R$ and the plane if rotation.
Then from the geometry one can deduce:
$$\theta = \phi - \alpha = \arctan \frac{V_0}{\Omega - \omega} = \arctan \frac{V_0(1 + a)}{\Omega(1 - a_1)}$$
or
$$\alpha = \theta - \arctan \frac{V_0 + v}{\Omega - \omega} \quad (1)$$

STATIONARY BLADE ELEMENT WITH AIR FLOWING PAST IT

The elemental lift expressed by the blade element is:
$$\partial L = \frac{1}{2} \rho V_R^2 C_l dr \quad (2)$$
Where $V_R^2 = (V + \omega)^2 + (r \ast (\Omega - \omega))^2$
The elemental drag is found to be:
$$\partial D = \frac{1}{2} \rho V_R^2 C_d dr \quad (3)$$
Where $C_l$ and $C_d$ are 2-D aerodynamic characteristics of the blade section.
From the force diagram
\[ \partial T = \partial L \cos \phi - \partial D \sin \phi \]  

(4)

If \( B \) is the number of blades the \( dT = b \partial T \).

i.e

\[ dT = bc \frac{1}{2} \rho V_r^2 (C_l \cos \phi - C_d \sin \phi) dr \]

(5)

Also from the force diagram: \( \frac{\partial Q}{r} = \partial L \sin \phi + \partial D \cos \phi \)

where \( \partial Q \) is the torque required to rotate the element about the axis of rotation.

Following the procedure used for \( dT \) one can show that the torque required to rotate \( B \) blades is given by:

\[ dQ = Bc \left( \frac{1}{2} \rho V_r^2 \right) (C_l \sin \phi + C_d \cos \phi) r dr \]

(6)

From \( dT \) and \( dQ \) one can obtain the thrust (T), torque (Q), and power required (P) using the equations given below:
\[ T = \int_0^R dT \quad (7) \]
\[ Q = \int_0^R dQ \quad (8) \]
\[ P = Q \Omega \quad (9) \]

\( \frac{dT}{dr} \) and \( \frac{dQ}{dr} \) are known as the thrust grading and the torque grading respectively.

Thrust \( dT \) from momentum principle: \( dT = \dot{m} \delta V = (\text{area of annulus} \times \text{velocity} \times \text{density}) \delta v \)

or

\[ dT = (2\pi r dr \times V(1 + a) \rho)(V_e - V_0) \]
\[ dT = (2\pi r dr \times V(1 + a) \rho)(v_0(1 + 2a) - V_0) \]
\[ = (2\pi r \rho V_0^2(1 + a)(2a) dr \quad (10) \]

Equating 5 and 10

\[ bc(\tfrac{1}{2} \rho V_0^2)(C_l \cos \phi - C_d \sin \phi) dr = 2\pi r \rho V_0^2(1 + a)(2a) dr \]
upon rearrangement:
\\[
\frac{bc}{4\pi r} V_R^2 (C_l \cos \phi - C_d \sin \phi) = V_0^2 (1 + a)(2a)
\]
(11)

but from the velocity diagram:
\\[V_R = V_0 (1 + a) 1/ \sin \phi \]
(12)

substitute (12) in (11) and rearrange to get:
\\[
\frac{bc}{8\pi r} 1/ \sin^2 \phi (C_l \cos \phi - C_d \sin \phi) = \frac{a}{1 + a}
\]
(13)

Relative angular velocity of the flow far upstream is $\Omega$
Relative angular velocity of the flow at the disc is $\Omega - a_1 \Omega$ or $\Omega - \omega$
Relative angular velocity of the flow far downstream is $\Omega - 2a_1 \Omega$ or $\Omega - 2\omega$.

Elemental torque in the annulus $dQ$ is equal to the angular momentum change per unit time in the annulus \( \dot{m}(\delta V_T) \cdot r \)

\[dQ = (2\pi r dr \rho V_0(1 + a)(2a_1 \Omega)(r) \]
(14)

Equating 6 and 14
\\[
bc \frac{r}{2 \rho V_R^2} (C_l \sin \phi + C_d \cos \phi) dr
= 4\pi^3 \rho V_0(1 + a)\Omega a_1 dr
\]
(15)

From the velocity diagram:
\\[V_R = \Omega r (1 - a_1) \sec \phi \]
(16)

Substitute 12 and 16 into equation 15 and rearrange to get
\\[
\frac{bc}{8\pi r} (1/ \sin \phi)(1/ \cos \phi)(C_l \sin \phi + C_d \cos \phi) = \frac{a_1}{1 - a_1}
\]
(17)

**Summary**

- Assume $a$ and $a_1$
- Calculate $V_0(1 + a)$ and $\Omega r (1 - a_1)$ the velocity in the axial and tangential direction at the plane of rotation.
- Calculate $\alpha$ and $\phi$ where $\phi = \arctan \frac{V_0 + \nu}{(\Omega - \omega) r}$ $\alpha = \theta - \phi$
- Calculate $C_l$ and $C_d$ from 2-D sectional characteristics of the airfoil used.
- Calculate new $a$ and $a_1$ using: $\frac{a}{\alpha} = \frac{bc}{8\pi r} 1/ \sin^2 \phi(C_l \cos \phi - C_d \sin \phi)$

\[\frac{a_1}{1 - a_1} = \frac{bc}{8\pi r} 1/ \cos \phi \sec \phi (C_l \sin \phi + C_d \cos \phi) \]
- if $a$ and $a_1$ are converged go to step seven otherwise go to step 2 with new values of $a$ and $a_1$. (For numerical convergence take an arithmetic mean of old $a$ and $a$)
- Calculate $T$, $Q$, $P$ and $C_T$, $C_Q$, and $C_P$