1. \( \mathbf{c}' \beta \) is estimable exactly when \( \mathbf{c} \in \mathbf{C}(\mathbf{X}') \). This occurs exactly when \( \mathbf{c} = \mathbf{P}_x \mathbf{c} \), that is when \( \mathbf{c} \) is its own projection onto \( \mathbf{C}(\mathbf{X}') \). Clearly, \( \mathbf{P}_x = \mathbf{X}'(\mathbf{XX}')^{-1} \mathbf{X} \). Use R and find this matrix for the situation of Problem 3 of HW 1. Then use this matrix and R to decide which of the following linear combinations of parameters are estimable in this example: 
\[
\tau_1, \mu + \tau_1, \mu + \tau_2, 2\mu + \tau_1 + \tau_2, \tau_1 - \tau_2, \text{ and } (\tau_1 - \tau_2) - (\tau_3 - \tau_4)
\]
For those that are estimable, find the \( 6 \times 1 \) row vector \( \mathbf{c}'(\mathbf{XX})^{-1} \mathbf{X}' \) that when multiplied by \( \mathbf{Y} \) produces the ordinary least squares estimate of \( \mathbf{c}' \beta \).

2. Twice now you've been asked to compute projection matrices in R. It seems like it would be helpful to “automate” this. Have a look at Chapter 10 of An Introduction to R. Write a function (call it, say, \( \text{project} \)) that for an input matrix produces a matrix that projects vectors onto the column space of the input matrix. Test your function by running it on both \( \mathbf{X} \) and \( \mathbf{X}' \) for the situation of Problem 3 of HW 1.

3. Consider the (non-full-rank) “effects model” for the \( 2 \times 2 \) factorial (with 2 observations per cell) called example d in lecture.
   a) Determine which of the parametric functions below are estimable.
   \[
   \alpha_1, \alpha_2, \alpha_1 - \alpha_2, \mu + \alpha_1 + \beta_1, \mu + \alpha_1 + \beta_1 + \alpha \beta_{11}, \alpha \beta_{11}, \alpha \beta_{12} - \alpha \beta_{21} - (\alpha \beta_{22} - \alpha \beta_{21})
   \]
   For those that are estimable, find the \( 8 \times 1 \) row vector \( \mathbf{c}'(\mathbf{XX})^{-1} \mathbf{X}' \) that when multiplied by \( \mathbf{Y} \) produces the ordinary least squares estimate of \( \mathbf{c}' \beta \).
   b) For the parameter vector \( \beta \) written in the order used in class, consider the hypothesis \( H_0: \mathbf{C} \beta = \mathbf{0} \), for
   \[
   \mathbf{C} = \begin{pmatrix}
   0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 1 & -1 & 1
   \end{pmatrix}
   \]
   Is this hypothesis “testable”? Explain.

4. Consider the one-variable quadratic regression of Problem 1 of HW1. Write the hypothesis that the values \( x \) and \( x' \) have the same mean response in the form \( H_0: \mathbf{C} \beta = \mathbf{d} \).

5. Suppose we are operating under the (common Gauss-Markov) assumptions that \( \mathbf{E} \epsilon = \mathbf{0} \) and \( \text{Var} \epsilon = \sigma^2 \mathbf{I} \).
   a) Use fact 1. of Appendix 7.1 of the class outline to find
   \[
   \mathbf{E}(\mathbf{Y} - \hat{\mathbf{Y}}) \text{ and } \text{Var}(\mathbf{Y} - \hat{\mathbf{Y}}). \text{ (Use the fact that } \mathbf{Y} - \hat{\mathbf{Y}} = (\mathbf{I} - \mathbf{P}_x) \mathbf{Y}).
   \]
   b) Then write
   \[
   \begin{pmatrix}
   \hat{\mathbf{Y}} \\
   \mathbf{Y} - \hat{\mathbf{Y}}
   \end{pmatrix} = \begin{pmatrix}
   \mathbf{P}_x \\
   \mathbf{I} - \mathbf{P}_x
   \end{pmatrix} \mathbf{Y}
   \]
and use fact 1 of Appendix 7.1 to argue that every entry of $Y - \hat{Y}$ is uncorrelated with every entry of $\hat{Y}$.

c) Use Theorem 5.2.A, page 95 of Rencher to argue that

$$E\left( Y - \hat{Y} \right) \left( Y - \hat{Y} \right)' = \sigma^2 (n - \text{rank} \left( X \right))$$

6. In the context of Problem 3 of HW 1 and the fake data vector used in Problem 4 of HW 1, use R and weighted generalized least squares to find appropriate estimates for

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1
\end{bmatrix} \beta
\]

in the Aitken models with

First $V_1 = \text{diag}(1, 4, 4, 1, 4)$ and then $V_2 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & -1 & 4
\end{bmatrix}$

For both of these covariance structures, compare the (Aitken model) covariance matrices for generalized least squares estimators to (Aitken model) covariance matrices for the OLS estimators of $EY$ and the $C\beta$ above.

7. In class Vardeman argued that hypotheses of the form $H_0: C\beta = 0$ can be written as $H_0: EY \in C(X_0)$ for $X_0$ a suitable matrix (and $C(X_0) \subset C(X)$). Let's investigate this notion in the context of Problem 3. Consider

\[
C = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\
0 & 1 & -1 & 0 & 0 & .5 & .5 & -.5 & -.5
\end{bmatrix}
\]

and the hypothesis $H_0: C\beta = 0$.

a) Find a matrix $A$ such that $C = AX$.

b) Let $X_0$ be the matrix consisting of the 1st, 4th and 5th columns of $X$. Argue that the hypothesis under consideration is equivalent to the hypothesis $H_0: EY \in C(X_0)$. (Note: One clearly has $C(X_0) \subset C(X)$. To show that $C(X_0) \subset C(A')^\perp$ it suffices to show that $P_X X_0 = 0$ and you can use R to do this. Then the dimension of $C(X_0)$ is clearly 2, i.e. rank($X_0$) = 2. So $C(X_0)$ is a subspace of $C(X) \cap C(A')^\perp$ of dimension 2. But the dimension of $C(X) \cap C(A')^\perp$ is itself rank($X$) - rank($C$) = 4 - 2 = 2.)

c) Interpret the null hypothesis under discussion here in Stat 500 language.