Some Notes on Elementary Statistical Considerations in Metrology

The basic Measurement Model (display (2.1), page 19 of V&J) is

\[ y = x + \epsilon \]

where \( x \) is a "true value" of interest, \( \epsilon \) is a measurement error, and \( y \) is what is actually observed. We assume that \( \epsilon \) is a random variable with mean \( \beta \) (the gauge "bias") and standard deviation \( \sigma_{\text{measurement}} \). (So under this model, repeat observation of the same \( x \) does not produce the same \( y \).) Under this model, with \( x \) fixed

\[ E_y = x + \beta \quad \text{and} \quad \text{Var}_y = \sigma_{\text{measurement}}^2 \]

On the other hand, with \( x \) random/varying and \( \epsilon \) independent of \( x \)

\[ E_y = \mu_x + \beta \quad \text{and} \quad \text{Var}_y = \sigma_x^2 + \sigma_{\text{measurement}}^2 \]

For a sample of \( m \) observations on the same unit with sample mean \( \bar{y} \) and sample standard deviation \( s \),

\[ E\bar{y} = x + \beta \quad \text{and} \quad E\bar{s}^2 = \sigma_{\text{measurement}}^2 \]

and basic statistical methods can be applied to \( \bar{y} \) and \( s \) to produce inferences of metrological interest. Consider first the usual confidence limits for a mean

\[ \bar{y} \pm t \frac{s}{\sqrt{m}} \]

(\( t \) is based on \( m - 1 \) degrees of freedom.) These are limits for \( x + \beta \). If the gauge is known to be well-calibrated (have 0 bias), they are limits for \( x \), the single true value for the unit being measured. On the other hand, if \( x \) is known because the unit being measured is a standard, it then follows that limits

\[ (\bar{y} - x) \pm t \frac{s}{\sqrt{m}} \]

can serve as confidence limits for the gauge bias, \( \beta \). Then consider the usual confidence limits for a standard deviation

\[ \left( s \sqrt{\frac{(m-1)}{\chi^2_{m-1,\text{upper}}}}, s \sqrt{\frac{(m-1)}{\chi^2_{m-1,\text{lower}}}} \right) \]

In the present context, these are limits for estimates \( \sigma_{\text{measurement}} \). Finally, mostly for purposes of comparison with other formulas, we might also note that a standard error for (an estimated standard deviation of) \( s \) is

\[ s \sqrt{\frac{1}{2(m-1)}} \]
For a sample of \( n \) observations, each on a different unit
\[
E[y] = \mu_x + \beta \quad \text{and} \quad E[s_y^2] = \sigma_x^2 + \sigma_{\text{measurement}}^2
\]
Applying the usual confidence limits for a mean,
\[
y \pm t \frac{s_y}{\sqrt{n}}
\]
(\( t \) is based on \( n - 1 \) degrees of freedom) are limits for \( \mu_x + \beta \), the mean of the distribution of true values for all units, plus bias. Note that the quantity \( s_y \) doesn’t directly estimate anything of fundamental interest. But since
\[
\sigma_x = \sqrt{(\sigma_x^2 + \sigma_{\text{measurement}}^2) - \sigma_{\text{measurement}}^2}
\]
an estimate of unit-to-unit variation (free of measurement noise) based on a sample of \( m \) observations on a single unit and a sample of \( n \) observations each on different units is (see display (2.3), page 20 of V&J):
\[
\hat{\sigma}_x = \max \left( 0, s_y^2 - s^2 \right) \quad (\#)
\]
The best currently available confidence limits on \( \sigma_x \) are complicated. But some reasonably elementary very approximate limits (based on what is known as “the Satterthwaite approximation”) can be made. These are
\[
\left( \hat{\sigma}_x \sqrt{\frac{\hat{\nu}}{\hat{\nu} \chi^2_{\hat{\nu} \text{upper}}}}, \hat{\sigma}_x \sqrt{\frac{\hat{\nu}}{\hat{\nu} \chi^2_{\hat{\nu} \text{lower}}}} \right)
\]
for
\[
\hat{\nu} = \frac{\hat{\sigma}_x^4}{s_y^4} \left( \frac{1}{n - 1} + \frac{1}{m - 1} \right)
\]
And it is possible to produce a “standard error” (an estimated standard deviation) for the estimate (*) as:
\[
\hat{\sigma}_x \sqrt{\frac{1}{2\hat{\nu}}}
\]
(we’re here ignoring the fact that these formulas can produce nonsense in the case that \( \hat{\sigma}_x = 0 \)). These approximate confidence limits and standard error give at least some feeling for how much one has really learned about \( \sigma_x \) based on the two samples.