Control Chart Construction

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1 Charts for Measurements

- **Rational subgroups**

Observed measurements are grouped together into rational subgroups. For each subgroup, the subgroup **Mean** (average) is calculated as well as a subgroup **Range** (difference between the largest and smallest values in a subgroup.) The subgroup ranges provide information about short-term (within subgroup) variability. The subgroup means, when viewed over time, provide information about longer-term (between subgroup) variability. Control limits are established based on the short-term variability (the ranges) and the average of the subgroup means. These are sometimes referred to as "retrospective" limits. The $\overline{X} - R$ control chart pair provides a pictorial representation of the short- and long-term variability in the process.

The following notation will be helpful in developing formulas for the control limits.

Data: X_{ti} is the i^{th} measurement in the t^{th} subgroup. There are n measurements in each subgroup[1] and m subgroups.

Range: $R_t = maximum(X_{ti}) - minimum(X_{ti})$ for t=1, 2, 3, ..., m

Average Range: $\overline{R} = \frac{\left(\sum R_t\right)}{m}$

Mean: $\overline{X}_t = \frac{\left(\sum_{i=1}^{n} X_{ti}\right)}{n}$

Average Mean: $\overline{\overline{X}} = \frac{\left(\sum \overline{X}_t\right)}{m}$

Estimate of short term standard deviation: $\hat{\sigma} = \frac{\overline{R}}{d_2}$.

Coefficients, such as d_2 and others used on the following page, can be found in Table 1 on page 2 of this handout. These coefficients are based on the assumption that measurements are normally distributed.

[1] The number of measurements in each subgroup does not have to be the same, but it is simpler to have a constant subgroup size.

– **Range (R) Chart**: Plot: R_t for t=1, 2, 3, ..., m.

UCL: $D_2\left(\frac{\overline{R}}{d_2}\right)=D_4\overline{R}$

CL: $\overline{R}$

LCL: $D_1\left(\frac{\overline{R}}{d_2}\right)=D_3\overline{R}$

– **Subgroup Mean ($\overline{X}$) Chart**: Plot: $\overline{X_t}$ for t=1, 2, 3, ..., m.

UCL: $\overline{\overline{X}}+3\frac{\left(\frac{\overline{R}}{d_2}\right)}{\sqrt{n}}=\overline{\overline{X}}+A_2\overline{R}$

CL: $\overline{\overline{X}}$

LCL: $\overline{\overline{X}}-3\frac{\left(\frac{\overline{R}}{d_2}\right)}{\sqrt{n}}=\overline{\overline{X}}-A_2\overline{R}$

Table 1: Control Chart Coefficients

Subgroup Size n	d_2	D_1	D_2	D_3	D_4	A_2
2	1.128	0	3.686	0	3.267	1.880
3	1.693	0	4.358	0	2.575	1.023
4	2.059	0	4.698	0	2.282	0.729
5	2.326	0	4.918	0	2.115	0.577
6	2.534	0	5.078	0	2.004	0.483
7	2.704	0.205	5.203	0.076	1.924	0.419
8	2.847	0.387	5.307	0.136	1.864	0.373
9	2.970	0.546	5.394	0.184	1.816	0.337
10	3.078	0.687	5.469	0.223	1.777	0.308
11	3.173	0.812	5.534	0.256	1.744	0.285
12	3.258	0.924	5.592	0.284	1.716	0.266
13	3.336	1.026	5.646	0.308	1.692	0.249
14	3.407	1.121	5.693	0.329	1.671	0.235
15	3.472	1.207	5.737	0.348	1.652	0.223
20	3.735	1.548	5.922	0.414	1.586	0.180
25	3.931	1.804	6.058	0.459	1.541	0.153

- **Rational Subgroups (Cont.)**

The subgroup range is not the only way to measure short-term variability. Another common measure is the subgroup **Standard Deviation**. The subgroup standard deviation replaces the subgroup range in the calculation of control limits. Appropriate coefficients can be found in Table 2 on page 4 of this handout. Again, the coefficients are based on the assumption that measurements are normally distributed.

The following notation will be helpful in developing formulas for the control limits.

Data: X_{ti} is the i^{th} measurement in the t^{th} subgroup. There are n measurements in each subgroup and m subgroups.

Standard Deviation:

$$s_t = \sqrt{\frac{\left[\sum_{i=1}^{n}\left(X_{ti} - \overline{X_t}\right)^2\right]}{(n-1)}}$$

or

$$s_t = \sqrt{\frac{\left[\sum_{i=1}^{n} X_{ti}^2 - \frac{\left(\sum_{i=1}^{n} X_{ti}\right)^2}{n}\right]}{(n-1)}}$$

for t=1, 2, 3, ..., m.

Average Standard Deviation: $\overline{s} = \frac{\sum s_t}{m}$

Mean: $\overline{X_t} = \frac{\left(\sum_{i=1}^{n} X_{ti}\right)}{n}$

Average Mean: $\overline{\overline{X}} = \frac{\left(\sum \overline{X_t}\right)}{m}$

Estimate of short term standard deviation: $\tilde{\sigma} = \frac{\overline{s}}{c_4}$.[2]

[2]There are other ways to estimate the short-term standard deviation based on the subgroup standard deviations. One alternative uses the square root of the pooled variance, s_p. Provided subgroup sizes are equal: $s_p = \sqrt{\frac{\sum s_t^2}{m}}$.

– **Standard Deviation (s) Chart**: Plot: s_t for t=1, 2, 3, ..., m.

UCL: $\overline{s} + 3\frac{\overline{s}}{c_4}\sqrt{1-c_4^2} = B_4\overline{s}$

CL: $\overline{s}$

UCL: $\overline{s} - 3\frac{\overline{s}}{c_4}\sqrt{1-c_4^2} = B_3\overline{s}$

– **Subgroup Mean ($\overline{X}$) Chart**: Plot: $\overline{X_t}$ for t=1, 2, 3, ..., m.

UCL: $\overline{\overline{X}} + 3\frac{\left(\frac{\overline{s}}{c_4}\right)}{\sqrt{n}} = \overline{\overline{X}} + A_3\overline{s}$

CL: $\overline{\overline{X}}$

LCL: $\overline{\overline{X}} - 3\frac{\left(\frac{\overline{s}}{c_4}\right)}{\sqrt{n}} = \overline{\overline{X}} - A_3\overline{s}$

Table 2: More Control Chart Coefficients

Subgroup Size n	c_4	B_3	B_4	B_5	B_6	A_3
2	0.7979	0	3.267	0	2.606	2.659
3	0.8862	0	2.568	0	2.276	1.954
4	0.9213	0	2.266	0	2.088	1.628
5	0.9400	0	2.089	0	1.964	1.427
6	0.9515	0.030	1.970	0.029	1.874	1.287
7	0.9594	0.118	1.882	0.113	1.806	1.182
8	0.9650	0.185	1.815	0.179	1.751	1.099
9	0.9693	0.239	1.761	0.232	1.707	1.032
10	0.9727	0.284	1.716	0.276	1.669	0.975
11	0.9754	0.321	1.679	0.313	1.637	0.927
12	0.9776	0.354	1.646	0.346	1.610	0.886
13	0.9794	0.382	1.618	0.374	1.585	0.850
14	0.9810	0.406	1.594	0.399	1.563	0.817
15	0.9823	0.428	1.572	0.421	1.544	0.789
20	0.9869	0.510	1.490	0.504	1.470	0.680
25	0.9896	0.565	1.435	0.559	1.420	0.606

- **Individual Measurements**

  Occasionally measurements are taken in such a way that no rational subgrouping is possible. In this case, each individual measurement becomes its own subgroup of size 1. With subgroups of size 1, short-term variability must be quantified differently. The **Moving Range**, the absolute difference of successive measurements, provides information about short-term variability provided successive measurements are not separated too much in time. These moving ranges are treated like ranges from subgroups of size 2, *i.e.* d_2=1.128.

  The following notation will be helpful in developing formulas for the control limits.

  Data: X_t is the measurement at time t.

  Moving Range: MR_t=$|X_t - X_{t-1}|$ for t=2, 3, ..., m

  Average Moving Range: $\overline{MR}$=$\frac{\left(\sum MR_t\right)}{m-1}$

  Average Value: $\overline{X}$=$\frac{\left(\sum X_t\right)}{m}$

  Estimate of short term standard deviation: $\hat{\sigma}$=$\frac{\overline{MR}}{1.128}$.

  - **Moving Range (MR) Chart**: Plot: MR_t for t=2, 3, ..., m.

    UCL: $3.267\overline{MR}$

    CL: $\overline{MR}$

    LCL: 0

  - **Individual Measurement (X) Chart**: Plot: X_t for t=1, 2, 3, ..., m.

    UCL: $\overline{X} + 2.660\overline{MR}$

    CL: $\overline{X}$

    LCL: $\overline{X} - 2.660\overline{MR}$

2 Charts for Counts and Proportions

- **Number of defective items in a subgroup of size** n

  When items in subgroups of size n are classified as either "good", *i.e.* conforming, nondefective, in-spec, *etc.* or "bad", *i.e.* nonconforming, defective, out-of-spec, *etc.*, control charts for the **Number of Defective Items** (or the **Fraction Defective**) in each subgroup can be constructed. The control limits are based on the normal approximation to the binomial distribution.

  The following notation will be helpful in developing formulas for the control limits.

  Data: Each subgroup consists of n items.[3] X_t is the **Number of Defective Items** in the t^{th} subgroup.

  Fraction Defective: $\hat{p}_t = \frac{X_t}{n}$ for t=1, 2, 3, ..., m.

  Average Fraction Defective: $\overline{p} = \frac{\sum X_t}{mn} = \frac{\sum \hat{p}_t}{m}$

  Estimate of the standard deviation of the fraction defective: $\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}$

  Estimate of the standard deviation of the number of defective items: $\sqrt{n\overline{p}(1-\overline{p})}$

  - p **Chart for fraction defective**: Plot: $\hat{p}_t$ for t=1, 2, 3, ..., m.

    UCL: $\overline{p} + 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}$

    CL: $\overline{p}$

    LCL: $\overline{p} - 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}$

  - np **Chart for number of defective items**: Plot: X_t for t=1, 2, 3, ..., m.

    UCL: $n\overline{p} + 3\sqrt{n\overline{p}(1-\overline{p})}$

    CL: $n\overline{p}$

    LCL: $n\overline{p} - 3\sqrt{n\overline{p}(1-\overline{p})}$

- **Number of defects per unit**

[3] Again, the number of items in a subgroup does not have to be the same, but it is simpler to have a constant subgroup size.

Rather than classify items as either "good" *i.e.* nondefective, or "bad" *i.e.* defective, one can count the number of defects per piece of standard size. Control charts can be constructed for the **Number of Defects**, or when pieces are of different sizes, the **Rate of Defects**. Control limits are based on the normal approximation to the Poisson distribution.

The following notation will be helpful in developing formulas for the control limits.

Data: Pieces of standard size are inspected and the number of defects, or imperfections, or nonconformities, are counted. c_t is the **Number of Defects** for the t^{th} piece. If pieces are of different sizes, let k_t denote the size of the t^{th} piece.

Rate of Defects: $u_t = \frac{c_t}{k_t}$

Average Number of Defects: $\overline{c} = \frac{\sum c_t}{m}$ (for the case where k_t is constant)

Averate Rate of Defects: $\overline{u} = \frac{\left(\sum c_t\right)}{\left(\sum k_t\right)}$

- c **Chart for Number of Defects** Plot: c_t for t=1, 2, 3, ..., m.

  UCL: $\overline{c} + 3\sqrt{\overline{c}}$

  CL: $\overline{c}$

  LCL: $\overline{c} - 3\sqrt{\overline{c}}$

- u **Chart for Rate of Defects** Plot: u_t for t=1, 2, 3, ..., m.

  UCL: $\overline{u} + 3\sqrt{\frac{\overline{u}}{k_t}}$

  CL: $\overline{u}$

  LCL: $\overline{u} - 3\sqrt{\frac{\overline{u}}{k_t}}$