Discrete Random Variables

General:

\[ p_X(x) = \Pr(X = x) \]

1. \( p_X(x) \geq 0 \)
2. \( \sum p_X(x) = 1 \)

\[ E(X) = \sum x p_X(x) = \mu \]
\[ \text{Var}(X) = \sum (x - \mu)^2 p_X(x) = \sigma^2 \]
\[ \text{Var}(X) = E(X^2) - \mu^2 \]
\[ \text{StdDev}(X) = \sqrt{\text{Var}(X)} = \sigma \]

Uniform: Select a ball at random from \( N \) balls labeled 1, 2, 3, \ldots, \( N \). Let \( X \) be the number on the ball selected.

\[ p_X(x) = \Pr(X = x) = \frac{1}{N} \quad x = 1, 2, 3, \ldots, N \]

\[ E(X) = \frac{N + 1}{2} \]
\[ \text{Var}(X) = \frac{(N - 1)(N + 1)}{12} \]

Bernoulli: A single Bernoulli trial is a success with probability \( p \) and a failure with probability \( (1 - p) \). Let \( X \) be 1 if the trial is a success and 0 if the trial is a failure.

\[ \Pr(X = 0) = 1 - p \]
\[ \Pr(X = 1) = p \]

\[ E(X) = p \]
\[ \text{Var}(X) = p(1 - p) \]
**Binomial:** Consider a sequence of \( n \) independent Bernoulli trials where the probability of success on any one trial is \( p \) and the probability of failure on any one trial is \((1 - p)\). Let \( Y \) be the number of successes.

\[
\Pr(Y = y) = \binom{n}{y} p^y (1 - p)^{n-y} \quad y = 0, 1, \ldots, n
\]

\[
E(Y) = np
\]

\[
\text{Var}(Y) = np(1 - p)
\]

**Geometric:** Consider a sequence of \( n \) independent Bernoulli trials where the probability of success on any one trial is \( p \) and the probability of failure on any one trial is \((1 - p)\). Let \( X \) be the number of trials until the first success. Note: Some texts define the geometric random variable \( Y \) as the number of the trial on which the first success occurs, \( Y = X + 1 \).

\[
\Pr(X = x) = (1 - p)^x p \quad x = 0, 1, 2, \ldots, n
\]

\[
\Pr(Y = y) = (1 - p)^{y-1} p \quad y = 1, 2, \ldots, n
\]

\[
E(X) = \frac{(1 - p)}{p}
\]

\[
E(Y) = \frac{1}{p}
\]

\[
\text{Var}(X) = \frac{(1 - p)}{p^2}
\]

\[
\text{Var}(Y) = \frac{(1 - p)}{p^2}
\]

**Negative Binomial:** Consider a sequence of \( n \) independent Bernoulli trials where the probability of success on any one trial is \( p \) and the probability of failure on any one trial is \((1 - p)\). Let \( Y \) be the number of the trial on which the \( r \)th success occurs.

\[
\Pr(Y = y) = \binom{y-1}{r-1} p^r (1 - p)^{y-r} \quad r = r, r + 1, r + 2, \ldots
\]

\[
E(Y) = \frac{r}{p}
\]

\[
\text{Var}(Y) = \frac{r(1 - p)}{p^2}
\]
Random sampling: An urn contains $N$ balls, $M$ of which are a particular color (green = success) and $N - M$ are a different color (red = failure). A sample of $n$ balls is drawn from the urn. Let $X$ be the number successes in the sample. If sampling is done with replacement $X$ is a Binomial random variable \( (n, p = \frac{M}{N}) \). If sampling is done without replacement $X$ is a Hypergeometric random variable \( (M, N, n) \).

Hypergeometric:

\[
\Pr(X = x) = \binom{M}{x} \binom{N-M}{n-x} \binom{N}{n}^{-1} \quad x = 0, 1, 2, \ldots, \min(n, M)
\]

\[
E(X) = n \left( \frac{M}{N} \right)
\]

\[
\text{Var}(X) = n \left( \frac{M}{N} \right) \left( 1 - \frac{M}{N} \right) \left( 1 - \frac{n-1}{N-1} \right)
\]

Poisson: Suppose rare events occur in a given time, area or volume at a rate given by $\lambda$. A Poisson random variable $X$ counts the number of rare events that occur in a given time, area or volume.

\[
\Pr(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad x = 0, 1, 2, \ldots
\]

\[
E(X) = \lambda
\]

\[
\text{Var}(X) = \lambda
\]