Chapter 12
Prediction of Future Random Quantities

William Q. Meeker and Luis A. Escobar
Iowa State University and Louisiana State University

Based on the authors' text

Chapter 12 Objectives

- Describe problem background and motivation, and some general prediction problem.
- Define probability prediction, naïve statistical prediction, and coverage probability.
- Discuss calibrating statistical prediction intervals and pivotal methods.
- Devise probability prediction, naïve statistical prediction, and some general prediction problem.
- Discuss problem background and motivation, and some

Introduction

Related Literature

New-Sample Prediction

Within-Sample Prediction

Extensions

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Iowa State University and Louisiana State University

Chapter 12
Prediction of Future Random Quantiles

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8h 9min

12-1

12-2

12-3

12-4

12-5

12-6
Unconditional coverage probability for the procedure •

In general to predict one needs:

For fixed DATA (and thus fixed

Random fields data.)

\( \text{Needed for Prediction} \)

\( \theta \) is an

\( \text{Objective: } \text{Want to predict the random quantity } \theta \text{ based on} \)

\( \text{Single future unit based on known parameters} \)

\( \text{Example 1: Probability Prediction for Future Time of} \)

\( \text{Nonparametric new-sample prediction also possible (e.g.,} \)

\( \text{Chapter 5 of Ham and Meeker 1998)} \)

\( \text{Field data} \)

\( \text{Laboratory test} \)

\( \text{Motion data comes from information on the values of the parameters } \theta. \text{ This infer-

\( \text{Errors } \theta \) is of interest. This model usually depends on a set of param-

\( \text{A statistical model to describe the population of process} \)

\( \text{In general to predict one needs:} \)

\( \text{Coverage Probabilistic Concepts} \)

\( \text{Statistical Prediction Interval} \)

\( \text{Joint distribution that depends on a parameter } \theta \)

\( \text{The random data lead to a parameter estimate and predic-

\( \text{a meaningful sample information (DATA)} \)

\( \text{Objective: } \text{Want to predict the random quantity } \theta \text{ based on} \)

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\( \text{A statistical model to describe the population of process} \)

\( \text{In general to predict one needs:} \)
To obtain a PI with a coverage probability of \(100(1-\alpha)\%\),

\[ CP\{(\hat{\theta}, \hat{\theta})|\hat{\theta}|\} = 1 - \alpha. \]

This gives the Monte Carlo estimate of the coverage probability when comparing with the simulated \(\hat{\theta}\) predictions of the \(g\) model. For each simulated data, use \(\alpha\) to compute \(\frac{\hat{\theta}}{g}\) from simulated \(\hat{\theta}\) data.

To simulate the sampling/prediction process, by computing \(\hat{\theta}\) from nominal \(\theta\) data.

For a two-sided interval, do separately for each tail.

To calibrate the naive one-sided prediction bound, find

\[ \hat{\theta}^\alpha \]

such that

\[ CP\{(\hat{\theta}, \hat{\theta})|\hat{\theta}|\} = 1 - \alpha. \]

For the \(\hat{\theta}\) cases, where \(\hat{\theta}\) is the ML estimator of the \(\theta\) parameter, and \(\hat{\theta}\) is the simulated \(\hat{\theta}\) data. Use the assumed model and all estimates to simulate.

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\[ \hat{\theta}^\alpha \]

such that

\[ CP\{(\hat{\theta}, \hat{\theta})|\hat{\theta}|\} = 1 - \alpha. \]
Lognormal probability plot of bearing life test data censored after 80 million cycles with lognormal ML estimates and pointwise 95% confidence intervals.

Simulation of the bearing life test censored after 80 million cycles ($n = 23$ and $r = 15$), lognormal model, histograms of pivotal–like $Z_{\log(T^*)} = (\log(T^*) - \hat{\mu}^*) / \hat{\sigma}^*$ and $\Phi[Z_{\log(T^*)}]^{-1}$.

Prediction interval calibration function for the bearing life test data censored after 80 million cycles, lognormal model.

0.93 0.95 0.97 0.99

1 - alpha_cal

CP[PI(1-alpha_cal);\theta^*] = 0.964

1-alpha_cal_lower = 0.967

1-alpha = 0.95

number of simulated samples = 100000

Lower Upper

Example 2: Lower Prediction Bound for a Single Independent Future $T$ Based on Time-Censored (Type I) Data

- Life test run for 80 million cycles; 15 of 23 ball bearings failed. ML estimates of the lognormal parameters are $\hat{\mu} = 4.160$, $\hat{\sigma} = 0.5451$.

- The naive one-sided lower 95% lognormal prediction bound (assuming no sampling error) is:

  $\hat{t}_{0.05} = \exp[4.160 + (0.1 - 1.645)(0.5451)] = 26.1$.

- Need to calibrate to account for sampling variability in the parameter estimates.

- From simulation CP[PI(1 - alpha_cal);\theta^*] = 0.95.

- Thus the calibrated lower 95% lognormal prediction bound is $T^* = \hat{t}_{0.036} = \exp[4.160 + (0.1 - 1.802)(0.5451)] = 24.0$.

Within-Sample Prediction

Predict Number of Failures in Next Time Interval

- The sample DATA are singly time-censored (Type I) from $F(t)$.

- Failure times are recorded for the $r > 0$ units that fail in $(0, t_c]$. Observe $n - r$ units until time $t_c$.

- Prediction problem: Find an upper bound for the number of failures $K$ in $(t_c, t_w]$, $t_c < t_w$.

- Prediction Interval Calibration Function for the bearing life test censored after 80 million cycles.

Comparison of 90% Prediction Intervals for Bearing Life from a Life Test that was Type I Censored at 80 Million Cycles

Lognormal

<table>
<thead>
<tr>
<th>Naive</th>
<th>Calibrated</th>
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</thead>
<tbody>
<tr>
<td>[26.1, 157.1]</td>
<td>[24.0, 174.4]</td>
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</tbody>
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<tr>
<th>12 - 22</th>
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<td>Predict Number of Failures in Next Time Interval</td>
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- Thus the calibrated lower 95% lognormal prediction bound is $T^* = \hat{t}_{0.036} = \exp[4.160 + (0.1 - 1.802)(0.5451)] = 24.0$.

- Where $\alpha_{cal} = 0.096 = 0.90^2$. 

- Thus the calibrated lower 95% lognormal prediction bound is:

  $\hat{t}_{0.05} = \exp[4.160 + (0.1 - 1.645)(0.5451)] = 26.1$.

- The naive one-sided lower 95% lognormal prediction bound is 4.160. $\theta^* = 0.545$.

<table>
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<td>Prediction Interval Calibration Function for the bearing life test censored after 80 million cycles.</td>
</tr>
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</table>
Distribution of $K$ and Naive Prediction Bound

• Conditional on DATA, the number of failures $K$ in $(t_{c}, t_{w}]$ is distributed as $K \sim \text{BIN}(n - r, \rho)$ where

\[
\rho = \Pr(t_{c} < T \leq t_{w}) / \Pr(T > t_{c}) = F(t_{w}; \theta) - F(t_{c}; \theta)
\]

- The naive 95% upper prediction bound for $K$ is $\tilde{K}(1 - \alpha) = \hat{K} 1 - \alpha$, the estimate of the $1 - \alpha$ quantile of the distribution of $K$. This is computed as the smallest integer such that

\[
\text{BINCDF}(K, n - r, \hat{\rho}) > 1 - \alpha.
\]

Example 3: Prediction of the Number of Future Failures

- $n = 10,000$ units put into service; 80 failures in 48 months.
- Weibull time to failure distribution assumed; ML estimates:

\[
\hat{\alpha} = 1152, \quad \hat{\beta} = 1.518
\]

- $\hat{\rho} = \hat{F}(60) - \hat{F}(48) / (1 - \hat{F}(48)) = 0.003233$.
- Point prediction for the number failing between 48 and 60 months is $n - r \times \hat{\rho} = 9920 \times 0.003233 = 32.07$.

• Similar for the lower prediction bound.

The naive 95% upper prediction bound on $K$ is $\tilde{K}(1 - \alpha) = 42$, the smallest integer $K$ such that $\text{BINCDF}(K, 9920, 0.003233) > 0.95$.

• From simulation $CP[PI(0.9863); \hat{\theta}] \approx 0.95$.

Example 3–Computations

- The naive 95% upper prediction bound on $K$ is $\tilde{K}(1 - \alpha) = 42$, the smallest integer $K$ such that $\text{BINCDF}(K, 9920, 0.003233) > 0.95$.

- From simulation $CP[PI(0.981); \hat{\theta}] = 0.986$.

Example 3: Calibration functions for upper and lower prediction bounds on the number of future field failures

- $CP[PI(1 - \alpha); \hat{\theta}] = 0.95$.

Example 3–Computations

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- From simulation $CP[PI(0.9863); \hat{\theta}] \approx 0.95$.

- Thus the calibrated 95% upper prediction bound on $K$ is $\tilde{K}(1 - \alpha) = 45$, the smallest integer $K$ such that $\text{BINCDF}(K, 9920, 0.003233) \geq 0.9863$.

- Weibull time to failure distribution assumed; ML estimates:

\[
\hat{\alpha} = 1152, \quad \hat{\beta} = 1.518
\]

- $\hat{\rho} = \hat{F}(60) - \hat{F}(48) / (1 - \hat{F}(48)) = 0.003233$.

- From simulation $CP[PI(0.986); \hat{\theta}] = 0.986$.

- Thus the calibrated 95% upper prediction bound on $K$ is $\tilde{K}(1 - \alpha) = 45$, the smallest integer $K$ such that $\text{BINCDF}(K, 9920, 0.003233) \geq 0.9863$.

Example 3: Calibration of the Naive Upper-Prediction Bound for the Number of Field Failures

- Find $\alpha_{c}$ such that $CP[PI(\alpha_{c}); \hat{\theta}] = 1 - \alpha$.

- A Monte Carlo evaluation of the unconditional coverage probability is

\[
CP[PI(\alpha_{c}); \hat{\theta}] = 1 - \alpha_{c} = \frac{1}{B} \sum_{j=1}^{B} P_{j}
\]

where

\[
P_{j} = \text{BINCDF}(K \tilde{\alpha}^{j}; n - r \tilde{\alpha}^{j}, \hat{\rho})
\]

is the conditional coverage probability for the $j$th simulated interval evaluated at $\hat{\rho}$.

- Similar for the lower prediction bound.
12-32

BEARING-CAGE FIELD-FAILURE DATA

<table>
<thead>
<tr>
<th>Group</th>
<th>Hours of Service</th>
<th>Failed</th>
<th>At Risk</th>
<th>Total</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>106</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>114</td>
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<tr>
<td>3</td>
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<tr>
<td>4</td>
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<td>9</td>
<td>506</td>
<td>1</td>
<td>1</td>
<td>50</td>
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failure units are observed in an additional period of length (t)

Within-sample prediction with staggered entry

Staggered Entry Prediction Problem

12-33

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BEARING-CAGE FIELD-FAILURE DATA (from Abernethy et al. 1983)

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Example 4: Calibration functions for upper and lower prediction bounds on the number of future field failures with staggered entry.

\[ CP[\hat{\alpha}; \hat{\theta}] \]

\[ CP[1-\alpha; \hat{\theta}] \]

\[ 1 - \alpha_{\text{cal}} \]

\[ 1 - \alpha = 0.95 \]

\[ \text{number of simulated samples} = 100000 \]

Example 4–Computations

• The naive 95% upper prediction bound on \( K \) is \( \hat{K}_{0.95} = 9 \), the smallest integer such that \( \text{SBINCDF}(K, n - r, \hat{\rho}) > 0.95 \).

• From simulation \( CP\{0.9916; \hat{\theta}\} \approx 0.95 \).

• Thus the calibrated 95% upper prediction bound on \( K \) is \( \hat{K}_{0.9916} = 11 \), the smallest integer such that \( \text{SBINCDF}(K, n - r, \hat{\rho}) > 0.9916 \).

The naive 95% upper prediction bound on \( K \) is \( \hat{K}_{0.95} = 9 \).

Concluding Remarks and Future Work

• Methodology can be extended to:
  - Staggered entry with differences among cohort distributions.
  - Staggered entry with differences in remaining warranty limits.
  - Staggered entry with differences among cohort distributions.
  - Modeling of spatial and temporal variability in environmental factors like UV radiation, acid rain, temperature, and humidity.

Today, the computational price is small; general-purpose software needed.

Asymptotic theory promises good approximation when not exact; use simulation to verify and compare with other approximations.

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