

- Describe the properties and the importance of the following parametric distributions which cannot be transformed into a location-scale distribution:

Gamma, Generalized Gamma, Extended Generalized Gamma, Generalized F, Inverse Gaussian, Birnbaum–Saunders, Gompertz–Makeham.

- Introduce the concept of a threshold-parameter distribution.

- Illustrate how other statistical models can be determined by applying basic ideas of probability theory to physical properties of a failure process, system, or population of units.

- Describe the properties and the importance of the following parametric distributions which cannot be transformed into a location-scale distribution:

Gamma, Generalized Gamma, Extended Generalized Gamma, Generalized F, Inverse Gaussian, Birnbaum–Saunders, Gompertz–Makeham.

- Introduce the concept of a threshold-parameter distribution.

- Illustrate how other statistical models can be determined by applying basic ideas of probability theory to physical properties of a failure process, system, or population of units.

- Describe the properties and the importance of the following parametric distributions which cannot be transformed into a location-scale distribution:

Gamma, Generalized Gamma, Extended Generalized Gamma, Generalized F, Inverse Gaussian, Birnbaum–Saunders, Gompertz–Makeham.

- Introduce the concept of a threshold-parameter distribution.

- Illustrate how other statistical models can be determined by applying basic ideas of probability theory to physical properties of a failure process, system, or population of units.

William Q. Meeker and Luis A. Escobar

Iowa State University and Louisiana State University

Copyright 1998-2008 W. Q. Meeker and L. A. Escobar.

Based on the authors' text *Statistical Methods for Reliability Data*, John Wiley & Sons Inc. 1998.

July 30, 2009

8h 3min

5-1

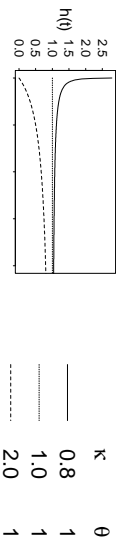
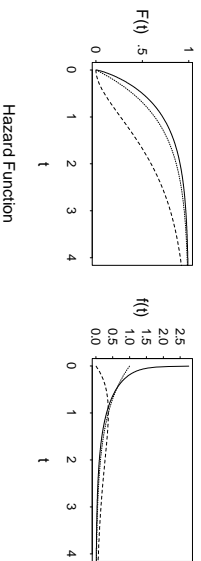
5-2

Examples of Gamma Distributions

Gamma Distribution

Cumulative Distribution Function

Probability Density Function



5-3

5-4

- T follows a gamma distribution, $\text{GAM}(\theta, \kappa)$, if

$$F(t; \theta, \kappa) = \Gamma\left(\frac{t}{\theta}; \kappa\right)$$

$$f(t; \theta, \kappa) = \frac{1}{\Gamma(\kappa)\theta} \left(\frac{t}{\theta}\right)^{\kappa-1} \exp\left(-\frac{t}{\theta}\right), \quad t > 0$$

$\theta > 0$ is a scale parameter and $\kappa > 0$ is a shape parameter. $\Gamma(v; \kappa)$ is the incomplete gamma function defined by

$$\Gamma(v; \kappa) = \frac{\int_0^v x^{\kappa-1} \exp(-x) dx}{\Gamma(\kappa)}, \quad v \geq 0.$$

- Special case:** when $\kappa = 1$, $\text{GAM}(\theta, \kappa) \equiv \text{EXP}(\theta)$.

- The hazard function $h(t; \theta, \kappa)$ is **decreasing** when $\kappa < 1$; **increasing** when $\kappa > 1$; and **approaches a constant level** late in life i.e.,

$$\lim_{t \rightarrow \infty} h(t; \theta, \kappa) = 1/\theta.$$

Moments and Quantiles of the Gamma Distribution

Reparameterization of the Gamma Distribution

- Moments:** For integer $m > 0$

$$E(T^m) = \frac{\theta^m \Gamma(m + \kappa)}{\Gamma(\kappa)}.$$

Then

$$E(T) = \theta \kappa$$

$$\text{Var}(T) = \theta^2 \kappa$$

- Quantiles:** the p quantile of the distribution is given by

$$t_p = \theta \Gamma^{-1}(p; \kappa).$$

For accelerated time regression modeling, the cdf and pdf can be conveniently **reparameterized** as follows:

$$F(t; \theta, \kappa) = \Phi_{\text{lg}}[\log(t) - \mu; \kappa]$$

$$f(t; \theta, \kappa) = \frac{1}{t} \phi_{\text{lg}}[\log(t) - \mu; \kappa]$$

where $\mu = \log(\theta)$, Φ_{lg} and ϕ_{lg} are the cdf and pdf for the **standardized** loggamma variable $Z = \log(T/\theta) = \log(T) - \mu$.

$$\Phi_{\text{lg}}(z; \kappa) = \Gamma[\exp(z); \kappa]$$

$$\phi_{\text{lg}}(z; \kappa) = \frac{1}{\Gamma(\kappa)} \exp[\kappa z - \exp(z)].$$

5-5

5-6

Generalized Gamma Distribution

- T has a generalized gamma distribution if

$$F(t; \theta, \beta, \kappa) = \Gamma_1 \left[\left(\frac{t}{\theta} \right)^\beta; \kappa \right]$$

$$f(t; \theta, \beta, \kappa) = \frac{\beta}{\Gamma(\kappa)\theta} \left(\frac{t}{\theta} \right)^{\kappa\beta-1} \exp \left[- \left(\frac{t}{\theta} \right)^\beta \right], \quad t > 0$$

where $\theta > 0$ is a scale parameter, and $\kappa > 0, \beta > 0$ are shape parameters.

- If $\beta = 1$ the distribution becomes the GAMM(θ, κ) distribution.
- If $\kappa = 1$ the distribution becomes the WEIB(μ, σ), where $\mu = \log(\theta)$ and $\sigma = 1/\beta$.
- If $\beta = 1$ and $\kappa = 1$ the distribution becomes the EXP(θ) distribution.

5-7

Generalized Gamma Distribution-Continued

- A more convenient parameterization is given by $\mu = \log(\theta) + (\sigma/\lambda) \log(\lambda^{-2})$, $\lambda = 1/\sqrt{\kappa}$, and $\sigma = 1/(\beta\sqrt{\kappa})$, in which case, we write $T \sim \text{GENG}(\mu, \sigma, \lambda)$ and

$$F(t; \mu, \sigma, \lambda) = \Phi_{\text{ig}} [\lambda\omega + \log(\lambda^{-2}), \lambda^{-2}]$$

$$f(t; \mu, \sigma, \lambda) = \frac{\lambda}{\sigma t^2} \phi_{\text{ig}} [\lambda\omega + \log(\lambda^{-2}), \lambda^{-2}]$$

where $\omega = [\log(t) - \mu]/\sigma$, $-\infty < \mu < \infty$, $\sigma > 0$, and $\lambda > 0$.

- If $T \sim \text{GENG}(\mu, \sigma, \lambda)$ and $c > 0$ then $cT \sim \text{GENG}[\mu - \log(c), \lambda, \sigma]$.
- As $\lambda \rightarrow 0$, $T \sim \text{LOGNOR}(\mu, \sigma)$.
- Moments, quantiles, and other related distributions will follow as special cases of the more general extended generalized gamma distribution.

5-8

Extended Generalized Gamma Distribution

- T has an extended generalized gamma distribution, EGENG(μ, σ, λ), if

$$F(t; \mu, \sigma, \lambda) = \begin{cases} \Phi_{\text{ig}} [\lambda\omega + \log(\lambda^{-2}), \lambda^{-2}] & \text{if } \lambda > 0 \\ \Phi_{\text{nor}}(\omega) & \text{if } \lambda = 0 \\ 1 - \Phi_{\text{ig}} [\lambda\omega + \log(\lambda^{-2}), \lambda^{-2}] & \text{if } \lambda < 0 \end{cases}$$

$$f(t; \mu, \sigma, \lambda) = \begin{cases} \frac{|\lambda|}{\sigma t} \phi_{\text{ig}} [\lambda\omega + \log(\lambda^{-2}), \lambda^{-2}] & \text{if } \lambda \neq 0 \\ \frac{1}{\sigma t} \phi_{\text{nor}}(\omega) & \text{if } \lambda = 0 \end{cases}$$

where $\omega = [\log(t) - \mu]/\sigma$, $-\infty < \mu < \infty$, $\exp(\mu)$ is a scale parameter, $-\infty < \lambda < \infty$ and $\sigma > 0$ are shape parameters.

5-9

Comments on the EGENG Distribution

- The distribution at $\lambda = 0$ is defined by **continuity** (i.e., the limiting distribution when $\lambda \rightarrow 0$).
- If $T \sim \text{EGENG}(\mu, \sigma, \lambda)$ and $c > 0$ then $cT \sim \text{EGENG}[\mu - \log(c), \lambda, \sigma]$. Thus, $\exp(\mu)$ is a location-parameter for T .
- When $T \sim \text{EGENG}(\mu, \lambda, \sigma)$ then the distribution of $W = [\log(T) - \mu]/\sigma$ depends only on λ .
- Note that for each fixed λ , $\log(T)$ is location-scale (μ, σ) with a standardized location-scale distribution equal to the distribution of W .

5-10

Extended Generalized Gamma Distribution—Continued

- **Moments:** For integer m and $\lambda \neq 0$

$$E(T^m) = \begin{cases} \frac{\exp(m\mu) (\lambda^2)^{m\sigma/\lambda} \Gamma[\lambda^{-1}(m\sigma + \lambda^{-1})]}{\Gamma(\lambda^{-2})} & \text{if } m\lambda\sigma + 1 > 0 \\ \infty & \text{if } m\lambda\sigma + 1 \leq 0. \end{cases}$$

When $\lambda = 0$, the moments are

$$E(T^m) = \exp [m\mu + (1/2)(m\sigma)^2].$$

- Thus when the mean and the variance are finite and $\lambda \neq 0$,

$$E(T) = \frac{\theta \Gamma [\lambda^{-1}(\sigma + \lambda^{-1})]}{\Gamma(\lambda^{-2})}$$

$$\text{Var}(T) = \theta^2 \left[\frac{\Gamma [\lambda^{-1}(2\sigma + \lambda^{-1})]}{\Gamma(\lambda^{-2})} - \frac{\Gamma^2 [\lambda^{-1}(\sigma + \lambda^{-1})]}{\Gamma^2(\lambda^{-2})} \right].$$

- When $\lambda = 0$, $E(T) = \exp[\mu + (1/2)\sigma^2]$ and $\text{Var}(T) = \exp(2\mu + \sigma^2) \times [\exp(\sigma^2) - 1]$.

5-11

Quantiles of the EGENG Distribution

The EGENG quantiles are

$$\log(t_p) = \mu + \sigma \times \omega(p; \lambda)$$

where $\omega(p; \lambda)$ is the p quantile of the distribution of W ,

$$\omega(p; \lambda) = \begin{cases} \lambda^{-1} \log [\lambda^2 \Gamma_1^{-1}(p; \lambda^{-2})] & \text{if } \lambda > 0 \\ \Phi_{\text{nor}}^{-1}(p) & \text{if } \lambda = 0 \\ \lambda^{-1} \log [\lambda^2 \Gamma_1^{-1}(1 - p; \lambda^{-2})] & \text{if } \lambda < 0 \end{cases}$$

5-12

Distributions Related to EGENG

Special Cases:

- If $\lambda > 0$ then EGENG(μ, σ, λ) = GENGG(μ, σ, λ).
- if $\lambda = 1$, $T \sim$ WEIB(μ, σ).
- if $\lambda = 0$, $T \sim$ LOGNOR(μ, σ).
- if $\lambda = -1$, $1/T \sim$ WEIB($-\mu, \sigma$), [i.e., T has a reciprocal Weibull (or Fréchet distribution of maxima)].
- When $\lambda = \sigma^2$, $T \sim$ GAM(θ, κ), where $\theta = \lambda^2 \exp(\mu)$ and $\kappa = \lambda^{-2}$.
- When $\lambda = \sigma = 1$, $T \sim$ EXP(θ), where $\theta = \lambda^2 \exp(\mu)$.

5-13

Comment on EGENG(μ, σ, λ) Parameterization

- The (μ, σ, λ) parameterization is due to Farewell and Prentice (1977). Observe that

$$F[\exp(\mu); \mu, \sigma, \lambda] = \begin{cases} \Gamma(\lambda^{-2}; \lambda^{-2}) & \text{if } \lambda > 0 \\ .5 & \text{if } \lambda = 0 \\ 1 - \Gamma(\lambda^{-2}; \lambda^{-2}) & \text{if } \lambda < 0 \end{cases}$$

This value of $F[\exp(\mu); \mu, \sigma, \lambda]$, as a function of λ , is always in the interval [.5, 1). Thus $\exp(\mu)$ equals a quantile t_p with $p \geq .5$.
- The parameterization is stable when there is not much censoring. It tends to be unstable when there is heavy censoring.
- When there is heavy censoring a different parameterization is needed for ML estimation.

5-14

EGENG Stable Parameterization

- **Parameterization for Numerical Stability:** with $p_1 < p_2$, an stable parameterization can be obtained using two quantiles (t_{p_1}, t_{p_2}), and λ , i.e.,

$$\begin{aligned} \log(t_{p_1}) &= \mu + \sigma w(p_1, \lambda) \\ \log(t_{p_2}) &= \mu + \sigma w(p_2, \lambda) \end{aligned}$$

and solving for μ and σ ,

$$\begin{aligned} \mu &= \frac{\omega(p_2, \lambda) \times \log(t_{p_1}) - \omega(p_1, \lambda) \times \log(t_{p_2})}{\omega(p_2, \lambda) - \omega(p_1, \lambda)} \\ \sigma &= \frac{\log(t_{p_2}) - \log(t_{p_1})}{\omega(p_2, \lambda) - \omega(p_1, \lambda)}. \end{aligned}$$

5-15

Generalized F Distribution

T has a generalized F distribution with parameters (μ, σ, κ, r) , say GENGF(μ, σ, κ, r), if

$$\begin{aligned} F_T^*(t; \mu, \sigma, \kappa, r) &= \Phi_{\text{F}} \left[\frac{\log(t) - \mu}{\sigma}; \kappa, r \right] \\ f_T^*(t; \mu, \sigma, \kappa, r) &= \frac{1}{\sigma t} \phi_{\text{F}} \left[\frac{\log(t) - \mu}{\sigma}; \kappa, r \right], \quad t > 0 \end{aligned}$$

where

$$\phi_{\text{F}}(z; \kappa, r) = \frac{\Gamma(\kappa + r)}{\Gamma(\kappa) \Gamma(r)} \frac{(\kappa/r)^\kappa \exp(\kappa z)}{[1 + (\kappa/r) \exp(z)]^{\kappa+r}}$$

is the pdf of the central log F distribution with 2κ and $2r$ degrees of freedom and Φ_{F} is the corresponding cdf.

It follows that $\phi_{\text{F}}(z; \kappa, r)$ and $\Phi_{\text{F}}(z; \kappa, r)$ are the pdf and cdf of $Z = [\log(T) - \mu]/\sigma$.

$\exp(\mu)$ is a scale parameter and $\sigma > 0$, $\kappa > 0$, $r > 0$ are shape parameters.

5-16

Generalized F Distribution–Continued

- **Moments:** For integer $m \geq 0$,
- $$E(T^m) = \begin{cases} \frac{\exp(\mu) \Gamma(\kappa+m) \Gamma(\kappa-m)}{\Gamma(\kappa) \Gamma(r)} \left(\frac{t}{\kappa}\right)^{m\sigma}, & \text{if } r > m\sigma \\ \infty & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} E(T) &= \frac{\Gamma(\kappa + \sigma) \Gamma(r - \sigma)}{\Gamma(\kappa) \Gamma(r)} \exp(\mu) \left(\frac{t}{\kappa}\right)^\sigma \\ \text{Var}(T) &= \left\{ \frac{\Gamma(\kappa + 2\sigma) \Gamma(r - 2\sigma)}{\Gamma(\kappa) \Gamma(r)} - \frac{\Gamma^2(\kappa + \sigma) \Gamma^2(r - \sigma)}{\Gamma^2(\kappa) \Gamma^2(r)} \right\} \exp(2\mu) \left(\frac{t}{\kappa}\right)^{2\sigma} \end{aligned}$$

where $r > \sigma$ for the mean and $r > 2\sigma$ for the variance.

- **Quantiles:** The p quantile of the distribution is

$$t_p = \exp(\mu) \left[\mathcal{F}_{(p; 2\kappa, 2r)} \right]^\sigma$$

where $\mathcal{F}_{(p; 2\kappa, 2r)}$ is the p quantile of an F distribution with $(2\kappa, 2r)$ degrees of freedom.

The expression for t_p follows directly from the fact that $T = \exp(\mu) V^\sigma$ where V has an F distribution with $(2\kappa, 2r)$ degrees of freedom.

5-17

Generalized F Distribution–Special Cases

- $1/T \sim$ GENF($-\mu, \sigma, r, \kappa$).
- When $(\mu, \sigma) = (0, 1)$ then T follows an F distribution with 2κ numerator and $2r$ denominator degrees of freedom.
- When $(\kappa, r) = (1, 1)$, GENF(μ, σ, κ, r) \equiv LOGLOGIS(μ, σ).
- When $r \rightarrow \infty$, $T \sim$ GENG[$\exp(\mu)/\kappa^\sigma, 1/\sigma, \kappa$].
- When $(\kappa, r) = (1, \infty)$, $T \sim$ WEIB(μ, σ).
- When $\kappa = 1$, T follows a Burr type XII distribution with cdf

$$F(t; \mu, \sigma, r) = 1 - \frac{1}{\left[1 + \frac{1}{t} \left(\frac{t}{\theta}\right)^{\frac{1}{\sigma}}\right]^r}, \quad t > 0$$

where $r > 0$, $\sigma > 0$ are shape parameters, and $\theta = \exp(\mu)$ is a scale parameter.

- When $\kappa \rightarrow \infty$, and $r \rightarrow \infty$, $T \sim$ LOGNOR($\mu, \sigma \sqrt{(\kappa + r)/\kappa r}$).

5-18

Inverse Gaussian Distribution

- A common parameterization for the cdf of this distribution is (see Chhikara and Folks 1989) is

$$\Pr(T \leq t; \theta, \lambda) = \Phi_{\text{nor}} \left[\frac{(t - \theta)\sqrt{\lambda}}{\theta\sqrt{t}} \right] + \exp \left(\frac{2\lambda}{\theta} \right) \Phi_{\text{nor}} \left[-\frac{(t + \theta)\sqrt{\lambda}}{\theta\sqrt{t}} \right]$$

$t > 0$; $\theta > 0$ and $\lambda > 0$ are parameters in the same units of T .

- Wald (1947) derived this distribution as a limiting form for the distribution of sample size in sequential probability ratio test.

5-19

Inverse Gaussian Distribution–Origin

- The inverse Gaussian distribution was originally given by Schrödinger (1915) as the distribution of the first passage time in Brownian motion. The parameters θ and λ relate to the Brownian motion parameters as follows:

- Consider a Brownian process

$$B(t) = ct + dW(t), \quad t > 0$$

where c, d are constants and $W(t)$ is a Wiener process. Let T be the first passage time of a specified level b_0 , say

$$T = \inf \{t; B(t) \geq b_0\}.$$

Then

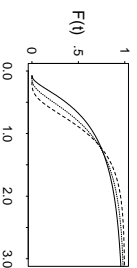
$$\Pr(T \leq t) = \Phi_{\text{nor}} \left[\frac{(t - \theta)\sqrt{\lambda}}{\theta\sqrt{t}} \right] + \exp \left(\frac{2\lambda}{\theta} \right) \Phi_{\text{nor}} \left[-\frac{(t + \theta)\sqrt{\lambda}}{\theta\sqrt{t}} \right]$$

where $\theta = b_0/c$ and $\sqrt{\lambda} = b_0/d$. Tweedie (1945) gives more details on this approach.

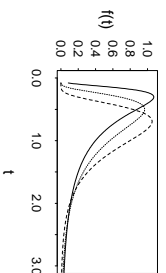
5-20

Examples of Inverse Gaussian Distributions

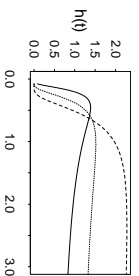
Cumulative Distribution Function



Probability Density Function



Hazard Function



β	θ
1	1
2	1
4	1

5-21

Inverse Gaussian Distribution–Continued

- The reparameterization $(\theta, \beta = \lambda/\theta)$ separates the location and scale parameters. We say that $T \sim \text{IGAU}(\theta, \beta)$ if

$$F_T(t; \theta, \beta) = \Phi_{\text{Ilgau}}[\log(t/\theta); \beta]$$

$$f_T(t; \theta, \beta) = \frac{1}{t} \phi_{\text{Ilgau}}[\log(t/\theta); \beta], \quad t > 0$$

where $\theta > 0$ is a scale parameter, $\beta > 0$ is at unit less shape parameter, and

$$\Phi_{\text{Ilgau}}(z; \beta) = \Phi_{\text{nor}} \left\{ \sqrt{\beta} \left[\frac{\exp(z) - 1}{\exp(z/2)} \right] \right\} + \exp(2\beta) \Phi_{\text{nor}} \left\{ -\sqrt{\beta} \left[\frac{\exp(z) + 1}{\exp(z/2)} \right] \right\}$$

$$\phi_{\text{Ilgau}}(z; \beta) = \frac{\sqrt{\beta}}{\exp(z/2)} \phi_{\text{nor}} \left\{ \sqrt{\beta} \left[\frac{\exp(z) - 1}{\exp(z/2)} \right] \right\}, \quad -\infty < z < \infty.$$

- The hazard function has the following behavior: $h_T(0; \theta, \beta) = 0$, $h_T(t; \theta, \beta)$ is unimodal, and $\lim_{t \rightarrow \infty} h_T(t; \theta, \beta) = \beta/(2\theta)$.

5-22

Inverse Gaussian Distribution–Continued

- Moments:** For integer $m > 0$

$$E(T^m) = \theta^m \sum_{i=0}^{m-1} \frac{(m-1+i)!}{i!(m-1-i)!} \left(\frac{1}{2\beta} \right)^i.$$

From this it follows that

$$E(T) = \theta \quad \text{and} \quad \text{Var}(T) = \theta^2/\beta.$$

- Quantiles:** the p quantile of the IGAU distribution is
- $$t_p = \theta \Phi_{\text{Ilgau}}^{-1}(p; \beta).$$

There is no simple closed form equation for $\Phi_{\text{Ilgau}}^{-1}(p; \beta)$, so it must be computed by inverting $p = \Phi_{\text{Ilgau}}(z; \beta)$ numerically.

5-23

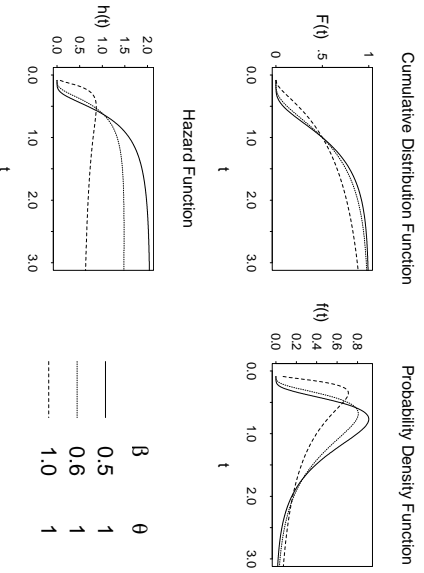
Inverse Gaussian Distribution–Continued

Special cases:

- If $T \sim \text{IGAU}(\theta, \beta)$ and $c > 0$ then $cT \sim \text{IGAU}(c\theta, \beta)$.
- For large values of β , the distribution is very similar to a $\text{NOR}(\theta, \theta/\sqrt{\beta})$.

5-24

Examples of Birnbaum–Saunders Distributions



5-25

Birnbaum–Saunders Distribution

- For a variable T with Birnbaum–Saunders distribution, $BISA(\theta, \beta)$,

$$F_T(t; \beta, \theta) = \Phi_{nor}(\zeta)$$

$$f_T(t; \beta, \theta) = \frac{\sqrt{\frac{t}{\theta}} + \sqrt{\frac{\theta}{t}}}{2\beta t} \phi_{nor}(\zeta)$$
 where $t \geq 0, \theta > 0$ is a scale parameter, $\beta > 0$ is a shape parameter, and

$$\zeta = \frac{1}{\beta} \left(\sqrt{\frac{t}{\theta}} - \sqrt{\frac{\theta}{t}} \right)$$

- **Moments:** For an integer $m > 0$,

$$E(T^m) = \theta^m \sum_{k=0}^m \beta^{2(m-k)} \frac{[2(m-k)!]}{[2^k(m-k)!]} \sum_{i=0}^{m-k} \binom{2m}{2k} \binom{m-k}{i}$$
 Then

$$E(T) = \theta \left(1 + \frac{\beta^2}{2} \right) \quad \text{and} \quad \text{Var}(T) = (\theta\beta)^2 \left(1 + \frac{5\beta^2}{4} \right).$$
- **Quantiles:** The p quantile is

$$t_p = \frac{\theta}{4} \left\{ \beta \Phi_{nor}^{-1}(p) + \sqrt{4 + [\beta \Phi_{nor}^{-1}(p)]^2} \right\}^2.$$

5-26

Birnbaum–Saunders Distribution–Continued

To isolate the scale parameter θ and the unitless shape parameter β , we write the cdf and pdf as follows

$$F_T(t; \beta, \theta) = \Phi_{bisa}[\log(t/\theta); \beta]$$

$$f_T(t; \beta, \theta) = \frac{1}{t} \phi_{bisa}[\log(t/\theta); \beta]$$

where

$$\Phi_{bisa}(z; \beta) = \Phi_{nor}(\nu)$$

$$\phi_{bisa}(z; \beta) = \left[\frac{\exp(z/2) + \exp(-z/2)}{2\beta} \right] \phi_{nor}(\nu), \quad -\infty < z < \infty$$

$$\nu = \frac{1}{\beta} [\exp(z/2) - \exp(-z/2)].$$

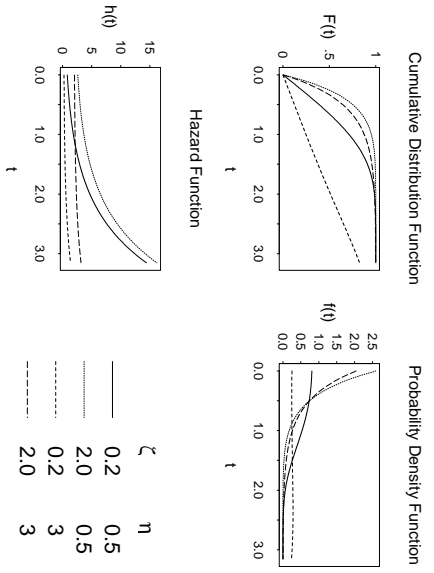
5-27

Birnbaum–Saunders Distribution–Continued

- Notes:**
- If $T \sim BISA(\theta, \beta)$ and $c > 0$ then $cT \sim BISA(c\theta, \beta)$.
 - If $T \sim BISA(\theta, \beta)$ then $1/T \sim BISA(\theta^{-1}, \beta)$.
 - The hazard function $BISA h(t; \theta, \beta)$ is not always increasing.
 - ▶ $h(0; \theta, \beta) = 0$.
 - ▶ $\lim_{t \rightarrow \infty} h(t; \theta, \beta) = 1/(2\theta\beta^2)$.
 - ▶ extensive numerical experiments indicate that $h(t; \theta, \beta)$ is always unimodal.
 - This distribution was derived by Birnbaum and Saunders (1969) in the modeling of fatigue crack extension.

5-28

Examples of Gompertz–Makeham Distributions



5-29

Gompertz–Makeham Distribution

- A common parameterization for this distribution is

$$\Pr(T \leq t; \gamma, \kappa, \lambda) = 1 - \exp \left[-\frac{\lambda \kappa t + \gamma \exp(\kappa t) - \gamma}{\kappa} \right], \quad t > 0.$$
 $\gamma > 0, \kappa > 0, \lambda \geq 0$ and all the parameters have units that are the reciprocal of the units of t .
- This distribution originated from the need of a positive random variable with a hazard function similar to the hazard of the SEV. It can be shown that

$$\Pr(T \leq t; \gamma, \kappa, \lambda) = 1 - \left[\frac{1 - \Phi_{sev} \left(\frac{t - \mu}{\sigma} \right)}{1 - \Phi_{sev} \left(\frac{-\mu}{\sigma} \right)} \right] \exp(-\lambda t)$$
 where $\mu = -(1/\kappa) \log(\gamma/\kappa)$, $\sigma = 1/\kappa$.
- When $\lambda = 0$, one gets Gompertz–distribution which corresponds to a truncated SEV at the origin.

5-30

Gompertz–Makeham Continued

The parameterization in terms of $[\theta, \psi, \eta] = [1/\kappa, \log(\kappa/\gamma), \lambda/\kappa]$ isolates the scale parameter from the shape parameter and we say that $T \sim \text{GOMA}(\theta, \psi, \eta)$, if

$$F_T(t; \theta, \psi, \eta) = \Phi_{\text{logoma}}[\log(t/\theta); \psi, \eta]$$

$$f_T(t; \theta, \psi, \eta) = \frac{1}{t} \phi_{\text{logoma}}[\log(t/\theta); \psi, \eta]$$

$$h_T(t; \theta, \psi, \eta) = \frac{\eta}{\theta} + \frac{\exp(-\psi)}{\theta} \exp\left(\frac{t}{\theta}\right), \quad t > 0$$

here θ is a scale parameter, ψ and η are unitless shape parameters, and

$$\Phi_{\text{logoma}}(z; \psi, \eta) = 1 - \exp\{\exp(-\psi) - \exp[\exp(z) - \psi] - \eta \exp(z)\}$$

$$\phi_{\text{logoma}}(z; \psi, \eta) = \exp(z) \{\eta + \exp[\exp(z) - \psi]\} [1 - \Phi_{\text{logoma}}(z; \psi, \eta)]$$

are, respectively, the standardized cdf and pdf of $Z = \log(t/\theta)$.

5-31

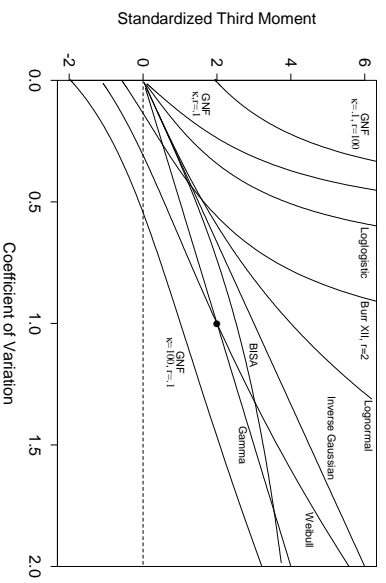
Gompertz–Makeham Distribution–Continued

Notes:

- $h_T(0; \theta, \psi, \eta) = (1/\theta)[\eta + \exp(-\psi)]$.
- $h_T(t; \theta, \psi, \eta)$ increases with t at an exponential rate.
- If $T \sim \text{GOMA}(\theta, \psi, \eta)$ and $c > 0$ then $cT \sim \text{GOMA}(c\theta, \psi, \eta)$.

5-32

Standardized Third Moment Versus Coefficient of Variation



5-33

Comparison of Spread and Skewness Parameters

- The **standardized** third central moment of T defined by

$$\gamma_3 = \frac{\int_0^\infty [t - E(T)]^3 f(t; \theta) dt}{[\text{Var}(T)]^{3/2}}$$

is a measure of the skewness in the distribution of T . This parameter is unitless and it has the these properties:

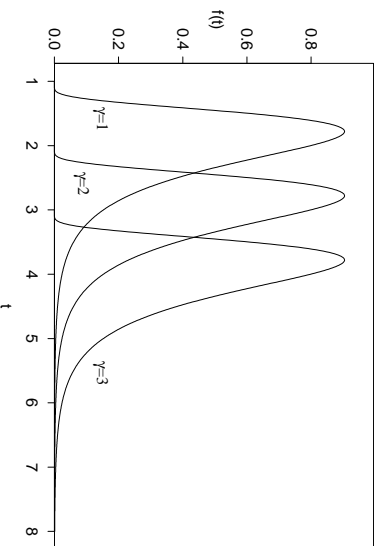
- ▶ Distributions with $\gamma_3 > 0$ will tend to be skewed to the right.
- ▶ Distributions with $\gamma_3 < 0$ will tend to be skewed to the left (e.g., the Weibull distribution with large β).

- The unitless **coefficient** of variation of T , $\gamma_2 = \sqrt{\text{Var}(T)}/E(T)$, is useful for comparing the relative amount of variability in the distributions of random variables having different units.

5-34

pdfs for Three-Parameter Lognormal Distributions for

$\mu = 0$ and $\sigma = .5$ with $\gamma = 1, 2, 3$.



5-35

Distributions with a Threshold Parameter

- So far we have discussed nonnegative random variables with cdfs that begin increasing at $t = 0$.
- One can generalize these and similar distributions by adding a **threshold**, γ , to shift the beginning of the distribution away from 0.
- Distributions with a threshold are particularly useful for fitting skewed distributions that are shifted far to the right of 0.
- The cdf for location-scale log-based threshold distributions is

$$F(t; \mu, \sigma, \gamma) = \Phi \left[\frac{\log(t - \gamma) - \mu}{\sigma} \right]$$

or $F(t; \eta, \sigma, \gamma) = \Phi \left[\log \left(\frac{t - \gamma}{\eta} \right) \right]^{1/\sigma}$, $t > \gamma$

where $\eta = \exp(\mu)$, $-\infty < \gamma < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$, $\eta > 0$, and Φ is a completely specified cdf.

5-36

Examples of Distributions with a Threshold Parameter

- Three-parameter lognormal distribution

$$F(t; \mu, \sigma, \gamma) = \Phi_{\text{nor}} \left[\frac{\log(t - \gamma) - \mu}{\sigma} \right], t > \gamma.$$

- Three-parameter Weibull distribution

$$\begin{aligned} F(t; \eta, \beta, \gamma) &= 1 - \exp \left[- \left(\frac{t - \gamma}{\eta} \right)^\beta \right] \\ &= \Phi_{\text{sev}} \left[\frac{\log(t - \gamma) - \mu}{\sigma} \right], t > \gamma \end{aligned}$$

where $\sigma = 1/\beta$ and $\mu = \log(\eta)$.

5-37

Properties of Distributions with a Threshold

- When the distribution of T has a threshold, γ , then the distribution of $W = T - \gamma$ has a distribution with 0 threshold.
- The properties of the distribution of T are **closely** related to the properties of the distribution of W .
- In general, $E(T) = \gamma + E(W)$ and $t_p = \gamma + w_p$, where w_p is the p quantile of the distribution of W .
- Changing γ simply shifts the distribution on the time axis, there is no effect on the distribution's spread or shape. Thus $\text{Var}(T) = \text{Var}(W)$.
- There are, however, some very specific issues in the estimation of γ because the points at which the cdf is positive depends on γ .

5-38

Embedded Models

- For some values of (μ, σ, γ) , the model is very similar to a two-parameter location-scale model, as described below.

- **Embedded models:** Using the **reparameterization**, $\alpha = \gamma + \eta$, $\varsigma = \sigma\eta$, the model becomes

$$\begin{aligned} F(t; \alpha, \sigma, \varsigma) &= \Phi \left[\log \left(1 + \sigma \times \frac{t - \alpha}{\varsigma} \right)^{1/\sigma} \right] \\ &= \Phi \left[\log (1 + \sigma z)^{1/\sigma} \right], \quad \text{for } z > -1/\sigma \end{aligned}$$

where $z = (t - \alpha)/\varsigma$.

When $\sigma \rightarrow 0^+$, $(1 + \sigma z)^{1/\sigma} \rightarrow \exp(z)$, and the **limiting** distribution is

$$F(t; \alpha, 0, \varsigma) = \Phi(z), \quad \text{for } -\infty < t < \infty.$$

- For example, if $\Phi = \Phi_{\text{sev}}$ the limiting distribution is the SEV and if $\Phi = \Phi_{\text{nor}}$ the limiting distribution is normal.

5-39

Some Comments on the Embedded Models

- The limiting distribution arises when
 - a. $1/\sigma$ and η are going to ∞ at the same rate, and
 - b. γ is going to $-\infty$ at the same rate that η is going to ∞ .
- Precisely, if $F(t; \eta_i, \sigma_i, \gamma_i)$ is a sequence of cdfs such that

$$\begin{aligned} \sigma_i &\rightarrow 0 \\ \varsigma &= \lim_{i \rightarrow \infty} (\sigma_i \eta_i) \quad \text{with } 0 < \varsigma < \infty \\ \alpha &= \lim_{i \rightarrow \infty} (\gamma_i + \eta_i) \quad \text{with } -\infty < \alpha < \infty \end{aligned}$$

then $F(t; \eta_i, \sigma_i, \gamma_i) \rightarrow \Phi(z)$, where $z = (t - \alpha)/\varsigma$

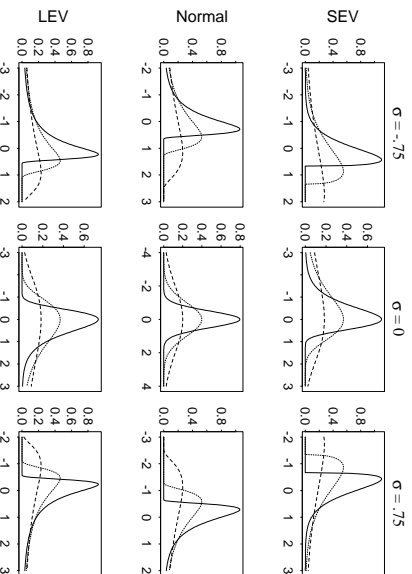
5-40

Generalized Threshold Scale (GETS) Models

- The original threshold parameter space $(\alpha, \sigma, \varsigma)$ (with $\sigma > 0$) does not contain the limiting distributions.
- It is convenient to enlarge the parameter space such that the limiting distributions are interior points of the parameter space.
- This is achieved by allowing σ to take values in $(-\infty, \infty)$.
- The family of distributions corresponding to this enlarged parameter space is known as the generalized threshold scale (GETS) family.

5-41

SEV-GETS, NOR-GETS, and LEV-GETS pdfs with $\alpha = 0$, $\sigma = -.75, 0, .75$, and $\varsigma = .5$ (Least Disperse), 1, and 2 (Most Disperse)



5-42

- The **cdf** for the GETS model is

$$F(t; \alpha, \sigma, \varsigma) = \begin{cases} \Phi \left[\log(1 + \sigma z)^{1/\sigma} \right], & \text{for } \sigma > 0, z > -1/\sigma \\ \Phi(z), & \text{for } \sigma = 0, -\infty < t < \infty \\ 1 - \Phi \left[\log(1 + \sigma z)^{1/\sigma} \right], & \text{for } \sigma < 0, z < -1/\sigma \end{cases}$$

where $z = (t - \alpha)/\varsigma$.

- The corresponding **pdf** is

$$f(t; \alpha, \sigma, \varsigma) = \begin{cases} \phi \left[\log(1 + \sigma z)^{1/\sigma} \right] \times \frac{1}{\varsigma(1 + \sigma z)}, & \text{for } \sigma \neq 0 \\ \phi(z) \times \frac{1}{\varsigma}, & \text{for } \sigma = 0, -\infty < t < \infty \end{cases}$$

Note: for $\sigma > 0, z > -1/\sigma$ and for $\sigma < 0, z < -1/\sigma$.

- If $T \sim \text{GETS}(\alpha, \sigma, \varsigma)$ and $a \neq 0$ then $(aT + b) \sim \text{GETS}(a\alpha + b, a\sigma/|a|, \varsigma|a|)$.

5-43

- The GETS model includes all the location-scales distributions. These are obtained when $\sigma = 0$, as

$$F(t; \alpha, 0, \varsigma) = \Phi[(t - \alpha)/\varsigma].$$

This includes the normal, logistic, SEV, LEV, etc.

- The GETS includes all the threshold, log-based location-scale distributions. These are obtained with $\sigma > 0$ which gives

$$F(t; \alpha, \sigma, \varsigma) = \Phi\{[\log(t - \gamma) - \mu]/\sigma\}, \quad t > \gamma$$

where $\gamma = \alpha - \varsigma/\sigma, \mu = \log(\varsigma/\sigma)$.

- With $\Phi = \Phi_{\text{nor}}$ this gives the lognormal with a threshold.

- With $\Phi = \Phi_{\text{sev}}$ this gives the Weibull (also known as Weibull-type for **minima**) with a threshold.

- And with $\Phi = \Phi_{\text{lev}}$ one obtains the Fréchet for **maxima** with a threshold.

5-44

Some Special Cases-Continued

- The GETS includes the reflection (negative) of the threshold, log-based location-scale distributions. These are obtained with $\sigma < 0$, giving

$$F(t; \alpha, \sigma, \varsigma) = \Phi\{[\log(-t - \gamma) - \mu]/\sigma\}, \quad t < -\gamma$$

where $\gamma = -(\alpha - \varsigma/\sigma), \mu = \log(-\varsigma/\sigma)$.

- With $\Phi = \Phi_{\text{nor}}$ this gives the negative of a lognormal with a threshold.

- With $\Phi = \Phi_{\text{sev}}$ this gives the negative of a Weibull with a threshold. Or equivalently a Weibull-type distribution for **maxima**.

- With with $\Phi = \Phi_{\text{lev}}$ one obtains the negative of a Fréchet for **maxima** with a threshold. Or equivalently, a Fréchet-type distribution for **minima**.

5-45

Quantiles for the GETS Distribution

- Quantiles:** the p quantile of the GETS distribution is

$$t_p = \alpha + \varsigma \times w(\alpha, p)$$

where

$$w(\alpha, p) = \begin{cases} \frac{\exp[\sigma \Phi^{-1}(p)] - 1}{\sigma}, & \text{for } \sigma > 0 \\ \Phi^{-1}(p), & \text{for } \sigma = 0 \\ \frac{\exp\{|\sigma| \Phi^{-1}(1-p)\} - 1}{\sigma}, & \text{for } \sigma < 0 \end{cases}$$

- Then for fixed σ, t_p versus $w(\alpha, p)$ plots as a straight line.

5-46

GETS Stable Parameterization

- Parameterization for Numerical Stability:** with $p_1 < p_2$, a stable parameterization can be obtained using two quantiles and $\sigma, i.e., (t_{p_1}, t_{p_2}, \sigma)$.

- Using the expression for the quantiles

$$\begin{aligned} t_{p_1} &= \alpha + \varsigma \times w(\sigma, p_1) \\ t_{p_2} &= \alpha + \varsigma \times w(\sigma, p_2). \end{aligned}$$

Solving for α and ς

$$\begin{aligned} \alpha &= \frac{w(\sigma, p_1) \times t_{p_2} - w(\sigma, p_2) \times t_{p_1}}{w(\sigma, p_1) - w(\sigma, p_2)} \\ \varsigma &= \frac{t_{p_1} - t_{p_2}}{w(\sigma, p_1) - w(\sigma, p_2)}. \end{aligned}$$

5-47

Finite (Discrete) Mixture Distributions

- The cdf of units in a population consisting of a mixture of units from k different populations can be expressed as

$$F(t; \theta) = \sum_i \xi_i F_i(t; \theta_i)$$

where $\theta = (\theta_1, \theta_2, \dots, \xi_1, \xi_2, \dots)$, $\xi_i \geq 0$, and $\sum_i \xi_i = 1$.

- Mixtures tend to have a large number of parameters and estimation can be complicated. But estimation is facilitated by:

- identification of the individual population from which sample units originated.
- considerable **separation** in the components and/or enormous amounts of data.

- Sometimes it is sufficient to fit a simpler distribution to describe the overall mixture.

5-48

Continuous Mixture (Compound Distributions)

- These probability models arise from distributions in which one or more of the parameters are continuous random variables.
- These distributions are called **compound** distributions and correspond to continuous mixture of a family of distributions, as follows:

Assume that for a fixed value of a scalar parameter θ_1 , $T|\theta_1 \sim f_{T|\theta_1}(t; \theta)$ with $\theta = (\theta_1, \theta_2)$. Assuming that θ_1 is random from unit to unit with $\theta_1 \sim f_{\theta_1}(\vartheta; \theta_3)$, where θ_3 does not have elements in common with θ , then

$$F(t; \theta_2, \theta_3) = \Pr(T \leq t) = \int_{-\infty}^{\infty} \Pr(T \leq t | \theta_1 = \vartheta) f_{\theta_1}(\vartheta; \theta_3) d\vartheta \\ = \int_{-\infty}^{\infty} F_{T|\theta_1=\vartheta}(t; \theta) f_{\theta_1}(\vartheta; \theta_3) d\vartheta$$

and the corresponding pdf is

$$f(t; \theta_2, \theta_3) = \int_{-\infty}^{\infty} f_{T|\theta_1=\vartheta}(t; \theta) f_{\theta_1}(\vartheta; \theta_3) d\vartheta.$$

5-49

Pareto Distribution as a Compound Distribution

- If life of the i th unit in a population can be modeled by $T|\eta \sim \text{EXP}(\eta)$.

- But the failure rate varies from unit to unit in the population according to a $\text{GAM}(\theta, \kappa)$, i.e.,

$$\frac{1}{\eta} \sim \text{GAM}(\theta, \kappa).$$

- Then the unconditional failure time of a unit selected at random from the population follows a Pareto distribution of the form

$$F(t; \theta, \kappa) = 1 - \frac{1}{(1 + \theta t)^\kappa}, \quad t > 0.$$

5-50

Other Distributions

- Power distributions.
- Distributions based on stochastic components of physical/chemical degradation models.
- Multivariate failure time distributions.

5-51