Chapter 4

Location-Scale-Based Parametric Distributions

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Objectives

• Explain importance of parametric models in the analysis of reliability data.
• Define important functions of model parameter that are of interest in reliability studies.
• Introduce the location-scale and log-location-scale families of distributions.
• Describe the properties of the exponential distribution.
• Describe the Weibull and lognormal distributions and the related underlying location-scale distributions.

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Motivation for Parametric Models

• Complements nonparametric techniques.
• Parametric models can be described concisely with just a few parameters, instead of having to report an entire curve.
• It is possible to use a parametric model to extrapolate (in time) to the lower or upper tail of a distribution.
• Parametric models provide smooth estimates of failure-time distributions.

In practice it is often useful to compare various parametric and nonparametric analyses of a data set.

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Functions of the Parameters

• Cumulative distribution function (cdf) of

\[ F(t; \theta) = \Pr(T \leq t) \quad t > 0. \]

• The q quantile of T is the smallest value \( t \) \( q \) such that

\[ F(t; \theta) \geq q. \]

• Hazard function of

\[ h(t; \theta) = \frac{f(t; \theta)}{1 - F(t; \theta)}, \quad t > 0. \]

• The mean time to failure, MTTF, of T (also known as the mean time to failure, MTTF, of T) does not exist.

• The variance (or the second central moment) of T and the standard deviation

\[ \Var(T) = \int_0^\infty [t - E(T)]^2 f(t; \theta) \, dt. \]

If \( \int_0^\infty t f(t; \theta) \, dt = \infty \), we say that the mean of T does not exist.

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Functions of the Parameters-Continued

• Coefficient of variation

\[ \gamma = \frac{SD(T)}{E(T)}. \]

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Location-Scale Distributions

Y belongs to the location-scale family of distributions if the cdf of Y can be expressed as

\[ F(y; \mu, \sigma) = \Phi \left( \frac{y - \mu}{\sigma} \right), \quad -\infty < y < \infty \]

where \( -\infty < \mu < \infty \) is a location parameter and \( \sigma > 0 \) is a scale parameter.

Note: The distribution of \( Z = \frac{y - \mu}{\sigma} \) does not depend on any unknown parameters.

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and nonparametric methods of a data set.

In practice it is often useful to compare various parametric distributions.

Parametric models provide smooth estimates of failure-time distribution
and nonparametric methods of a data set.

Parametric models can be described concisely with just a few parameters, instead of having to report an entire curve.

Parametric models can be developed concurrently with just a few parameters.

Complements nonparametric techniques.
**Exponential Distribution**

**Motivation:**
- Logical life of the system in which the component would be installed.
- Characteristic of electronic components having highly-non-linear behavior.
- Simplest distribution used in the analysis of reliability data.
- Has the important characteristic that its hazard function is constant (does not depend on time $t$).
- Probability density function for some kinds of electronic components.
- Equally useful to describe failure times for components (e.g., capacitors or robust, high-quality integrated circuits).

**Properties:**
- PDF: $f(t) = \lambda e^{-\lambda t}$ for $t > 0$.
- CDF: $F(t) = 1 - e^{-\lambda t}$ for $t > 0$.
- Hazard function: $h(t) = \lambda e^{-\lambda t}$ for $t > 0$.
- Mean: $E[T] = \frac{1}{\lambda}$.
- Variance: $Var(T) = \frac{1}{\lambda^2}$.

**Moments:**
- $E[T^m] = \frac{m!}{\lambda^m}$ for $m > 0$.

**Quantiles:**
- $F^{-1}(p) = \frac{-\ln(1-p)}{\lambda}$ for $0 < p < 1$.

**Important Distributions**
- Exponential is a location-scale distribution.
- It is widely used in reliability analysis.
- It is a simple and ideal distribution for modeling the time between events.

**Examples of Exponential Distributions**

![Examples of Exponential Distributions](image)
Useful in modeling failure time of a population electronic degradation processes.

Motivation for Lognormal Distribution

Examples of Weibull Distributions

Examples of smallest Extreme Value Distributions

Examples of lognormal Distributions

Moments:

Quantiles:

Quantiles:

Moments:

Quantiles:

Moments:

Quantiles:

Moments:
The theory of extreme values shows that the Weibull distribution can be used to model failure-time data where the hazard is increasing but is bounded in the sense that $\lim_{t \to \infty} h(t) = 0$. The more common justification for its use is empirical: the Weibull distribution can be used to model failure-time data with a decreasing or increasing hazard function. Weibull distribution can be used to model failure-time data with a decreasing or increasing hazard function.

**Examples of Largest Extreme Value Distributions**

- Standardized LEV: $F(t) = 1 - e^{-t^\beta}$, where $\beta > 0$ and $\eta > 0$ is a shape parameter.
- LEV: $F(t) = 1 - e^{-t^\beta - \eta t}$, where $\eta > 0$ and $\beta > 0$ are the cdf and pdf for a standardized LEV.

**Motivation for the Weibull Distribution**

When $\eta = 0$, the cdf and pdf of the standardized LEV are $F(t) = 1 - e^{-t^\beta}$ and $f(t) = \beta t^{\beta-1} e^{-t^\beta}$, respectively. Therefore, the Weibull distribution is a special case of the extreme value distribution. For integer $m > 0$, the $m$th moment of the Weibull distribution is given by $E(Y^m) = \eta^m (\beta + 1)$.
Examples of Logistic Distributions

\[ F(t) = \Phi_{\text{logis}} \left( \frac{t - \mu}{\sigma} \right) \]

where \( \Phi_{\text{logis}} \) is the cumulative distribution function for a standardized logistic distribution.

\[ f(t) = \frac{1}{\sigma} \phi_{\text{logis}} \left( \frac{t - \mu}{\sigma} \right) \]

\[ h(t) = \frac{1}{\sigma} \Phi_{\text{logis}} \left( \frac{t - \mu}{\sigma} \right) \phi \left( \frac{t - \mu}{\sigma} \right) \]

Quantiles:

\[ t_p = \exp \left( \mu + \sigma \Phi_{\text{logis}}^{-1}(p) \right) \]

Moments:

\[ E(T_m) = \exp(m\mu) \frac{\Gamma(1 + m\sigma)}{\Gamma(1 - m\sigma)} \]

For integer \( m > 0 \), the quantity is not finite when \( m \sigma \geq 1 \).

Examples of Loglogistic Distributions

\[ T = \exp(Y) \sim \text{LOGLOGIS}(\mu, \sigma) \]

with

\[ F(t) = \Phi_{\text{logis}} \left( \log(t) - \mu \sigma \right) \]

\[ f(t) = \frac{1}{\sigma t} \phi_{\text{logis}} \left( \log(t) - \mu \sigma \right) \phi \left( \log(t) - \mu \sigma \right) \]

\[ h(t) = \frac{1}{\sigma t} \Phi_{\text{logis}} \left( \log(t) - \mu \sigma \right) \phi \left( \log(t) - \mu \sigma \right) \]

Quantiles:

\[ t_p = \exp \left( \mu + \sigma \Phi_{\text{logis}}^{-1}(p) \right) \]

Moments:

\[ E(T^m) = \exp(m\mu) \frac{\Gamma(1 + m\sigma)}{\Gamma(1 - m\sigma)} \]

Logistic Distribution

The \( m \) moment is not finite when \( m \sigma \geq 1 \).

For \( \sigma < 1 \),

\[ E(T) = \exp(\mu) \frac{\Gamma(1 + \sigma)}{\Gamma(1 - \sigma)} \]

and for \( \sigma < 1/2 \),

\[ \text{Var}(T) = \exp(2\mu) \left[ \frac{\Gamma(1 + 2\sigma)}{\Gamma(1 - 2\sigma)} - \frac{\Gamma^2(1 + \sigma)}{\Gamma^2(1 - \sigma)} \right] \]

Loglogistic Distribution-Continued

Examples of Logistic Distributions

\[ \frac{\phi(X)}{\Phi(X)} = \frac{\phi_{\text{logis}} \left( \frac{X - \mu}{\sigma} \right)}{\Phi_{\text{logis}} \left( \frac{X - \mu}{\sigma} \right)} \]

where \( \phi \) and \( \Phi \) are pdf and cdf for a standard logistic distribution.

\[ \frac{\phi(X)}{\Phi(X)} = \frac{\phi_{\text{logis}} \left( \frac{X - \mu}{\sigma} \right)}{\Phi_{\text{logis}} \left( \frac{X - \mu}{\sigma} \right)} \]

Examples of Loglogistic Distributions

\[ \frac{\phi(X)}{\Phi(X)} = \frac{\phi_{\text{logis}} \left( \log(X) - \mu \sigma \right)}{\Phi_{\text{logis}} \left( \log(X) - \mu \sigma \right)} \]

Logistic Distribution

\[ \frac{\phi(X)}{\Phi(X)} = \frac{\phi_{\text{logis}} \left( \frac{X - \mu}{\sigma} \right)}{\Phi_{\text{logis}} \left( \frac{X - \mu}{\sigma} \right)} \]
Other Topics in Chapter 4

Pseudorandom number generation.

Topics in Chapter 5

• Parametric models with threshold parameters.

• Important distributions used in reliability that cannot be translated into location-scale distributions: gamma, generalized gamma, etc.

• Finite (discrete) mixture distributions

\[ F(t; \theta) = \sum_{i=1}^{k} \xi_i F_i(t; \theta_i) \]

where \( \xi_i \geq 0 \), and \( \sum_{i=1}^{k} \xi_i = 1 \)

• Compound (continuous) mixture distributions

• Mixtures of distributions: normal, gamma, etc.

• Binary mixture distribution

• Finite mixture distributions with threshold parameters.

• Importance sampling in reliability analysis.