Statistics 533
Spring 1998

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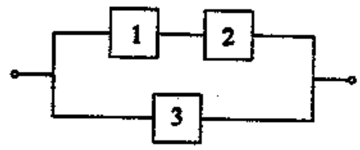
Final Examination

Name ________________

You must show all of your work

When asked to explain something, provide an explanation that could be understood by someone who does not have formal training in statistical methods. Your explanations should be clear, but concise. Each question or part of a question is worth 5 points for a total of 100 points.

1. Consider the following system diagram. Assuming that the three components fail independently, derive an expression for the cdf of the system as a function of F_1, F_2, and F_3, the cdfs of the individual components.



$$F_T(t) = \underbrace{F_3(t)}_{\text{parallel}}\Big[\underbrace{1-(1-F_1(t))(1-F_2(t))}_{\text{series}}\Big]$$

✓

2. Bayesian methods are becoming increasingly popular in many areas of application of statistics. Part of the reason for this is that modern computing capabilities have made it possible to solve real problems that could not be solved in the past. In the area of reliability, however, Bayesian methods are often viewed with some skepticism.

   (a) What is the most important advantage or potential advantage for the use of Bayesian methods in reliability data analysis and decision making?

   Bayesian methods allow us to use information from previous experience to augment the likelihood and increase precision in our estimates and conclusions ✓

   (b) What is the most important danger involved with the use of Bayesian methods in reliability data analysis and decision making?

   If our beliefs about the prior distribution are wrong, and particularly if $f(\theta)$ is large relative to $R(\theta)$, then the prior beliefs will overpower the data and may lead to seriously wrong conclusions, regardless of what the data says. ✓

   (c) Suppose that analysts have available prior information as well as Type I censored data from a sample of 100 switches. Given a Monte Carlo sample of 2000 pairs of μ^*, σ^* values from the posterior pdf $f(\mu, \sigma|\text{DATA})$, explain how to approximate the marginal posterior distribution for $t_{.1}$, the .1 quantile of the life distribution of the switches and how to find a 95% Bayesian confidence (or credible) interval for $t_{.1}$.

   ✓ Use the 2000 pairs of μ^*, σ^* values to calculate a $t_{.1}^*$ for each pair. This gives a simulation of the approximate marginal posterior dsn of $t_{.1}$. Then use this distribution to find 2 values of $t_{.1}$ such that 2.5% of the distribution falls in each tail (for equal tail interval) or use numerical methods to find the cutoff pt. for an HPD interval. .95 $t_{.1}^*$ dsn

3. A company that owns a fleet of 130 automobiles keeps track of all repairs and component replacements. Part of the data base provides the life times of components (e.g., light bulbs, hoses, and belts) that are replaced upon failure or through preventive maintenance in an attempt to reduce unexpected failures. For the purposes of this question, let us consider the bulbs used in head lights. These bulbs, at present, are not replaced until there is a failure.

(a) How would you interpret the mean cumulative function (MCF) for light bulbs failures estimated from information collected on this fleet of automobiles?

The MCF here is the number of failed headlights in any car in the fleet that one would expect to find on average, up to a given time, t_K. It is the average over 130 cars of each cumulative # of failures. ✓

(b) How would you interpret the pointwise confidence intervals for the MCF if they were computed from information collected on this fleet of automobiles? Hint: You have to specify the population or process being described by the confidence interval.

I am (95%) confident that among ~~same-sized samples~~ of similar cars, the true average headlight failures per car up to time t_K would fall in the interval. −1

(c) Suppose that the failure times of light bulbs within a car over time are identically and independently distributed. Then a renewal process would provide an appropriate model and it would make sense to try to estimate the lifetime distribution of the bulbs. Give some physical reasons why this assumption might *not* be valid.

✓ Light bulbs might not be identically distributed because of differences among the cars (ie. if a car drives rougher and might cause a wire to come loose) and they may not be independent because one would expect 2 headlights in same car to have same use conditions + similar lifetimes.

(d) If light bulbs failure times are identically and independently distributed, what shape would you expect for the hazard function of the light bulbs? Draw a picture and explain.

I would expect lightbulbs to have a nondecreasing hazard function (since there's pretty much a finite life time). I'd expect the hazard function to start off concave, increasing very slowly and then become convex near the mean of the distribution where the bulk of the failures occur (since light bulbs probably have little variance). ✓

(e) If failure times of any particular component in the automobiles are identically and independently distributed, how could the shape of the component's hazard function be used to indicate whether the component in an automobile should be replaced before it fails or not? Draw a picture and explain.

Looking at the hazard function of the component, one can tell how likely it is to fail in the next interval of time if it has survived thus far. So, for example, if the hazard function has a point at which it increases rapidly ex

it would be worthwhile to replace the component before failure ✓ because it is highly likely to fail shortly (and may leave you stranded!). If, on the other hand, the hazard is relatively flat or slowly increases (possibly decreases, though not often in practice) it is hard to tell when to replace it.

4. Refer to Table C.20 (copy attached). A life test is planned initially to run for 1000 hours and the responsible engineers feel that life can be described by a lognormal distribution with $\sigma = .5$ and with approximately 2.28% of the units failing at the end of the 1000 hour test.

(a) Management has asked for an estimate and a 95% confidence interval for $t_{.5}$ for this distribution. In planning the test, it is thought that the sample size should be large enough such that the confidence interval upper endpoint should be approximately 40% larger than the ML estimate of $t_{.5}$. What is the approximate sample size that is needed to estimate $t_{.5}$ for this distribution, according to the specified criterion, assuming that both μ and σ will be estimated using the method of maximum likelihood?

$$V^{\square}_{\log \hat{t}_p} = \sigma^2 (159.661) + (0)(\sigma^2\, 35.7402) + 2(0)\sigma^2\, 73.7498$$
$$= 159.661\sigma^2$$

$$n = \frac{z^2\, V^{\square}_{\log \hat{t}_p}}{(\log(R_T))^2} = \frac{(1.96)^2 (159.661\sigma^2)}{.1132} = \boxed{5417.67\sigma^2}$$

(OR 1355 if we let $\sigma = .5$)

(b) Figure 10.6 (copy attached) can be used to obtain the variance factor used in the equation to find the approximate sample size needed to estimate a particular lognormal quantile. Use this figure to find the sample size needed to estimate $t_{.1}$ such that a 95% confidence interval upper endpoint should be approximately 40% larger than the ML estimate of $t_{.1}$.

$$V^{\square}_{\log \hat{t}_p} \approx 30\sigma^2$$

$$n = \frac{(1.96)^2 (30\sigma^2)}{.1132} = \boxed{1018.09\sigma^2}$$

(255 if we let $\sigma = .5$)

(c) Give an equation to show how the variance factor from Figure 10.6, used in part (b), could be computed from the information in Table C.20.

$$V^{\square}_{\log \hat{t}_{.1}} = V_{\hat{\mu}} + [\Phi^{-1}(.1)]^2 V_{\hat{\sigma}} + 2\Phi^{-1}(.1)\, V_{\hat{\mu},\hat{\sigma}} =$$
$$= 159.661\sigma^2 + (-1.28)^2 35.7402\sigma^2 + 2(-1.28)(73.7498\sigma^2)$$
$$= 29.4183 \approx 30$$

(d) Note that some of the lines at the bottom of Figure 10.6 tend to bunch together. What is the intuitive explanation for this behavior of these lines? What general implication does this have for test planning?

("point of diminishing returns")

At the bottom, the "censoring proportion" approaches one. At that point the variance factors slowly approach one so that $\sigma^2 \times$(var factor) approaches σ^2. As we let units run closer to failure for all, the Var(t_p) approaches σ^2 slowly. This implies that for only a slightly larger sample size we can reduce test length a lot.

(e) Suppose that there is some flexibility in choosing the length of the test (i.e., it might be possible to test longer than 1000 hours, if needed) that is to be used to estimate $t_{.1}$. Use Figure 10.6 to make a qualitative argument as to whether this is a useful way to spend additional testing resources. How much could the needed sample size be reduced by making the test longer?

Since the contours are still fairly far apart, it might be worthwhile to increase test length somewhat. For example, if we increased the proportion failing from 2.28 to 3% (by lengthening the test) n would go from $1018\sigma^2$ to $678.7\sigma^2$, which one might consider quite substantial (or 255 to 170 if $\sigma = .5$, a 1/3 reduction)

good

5. The inverse power relationship-lognormal model is

$$\Pr[T \le t; \mathbf{volt}] = \Phi_{\text{nor}}\left[\frac{\log(t)-\mu}{\sigma}\right]$$

where $\mu = \beta_0 + \beta_1 x$, $x = \log(\mathbf{volt})$, $\mathbf{volt}$ is voltage stress that could be measured, for example, in kV/mm, and σ is constant.

(a) Derive an expression for the quantile of the lognormal distribution as a function of voltage stress.

$$t_p = \exp(\mu + \sigma z_p)$$
$$= \exp(\beta_0 + \beta_1 \log(\text{volt}) + \sigma\, \Phi^{-1}(p))$$
$$= e^{\beta_0} (\text{volt})^{\beta_1}\, e^{\Phi^{-1}(p)\sigma}$$

(b) Engineers often think in terms of acceleration factors. Derive an expression for the acceleration factor that one would have for testing at $\mathbf{volt}_1$ versus $\mathbf{volt}_2$.

$u=2$

$$T(\text{volt}_1) = \frac{T(\text{volt}_2)}{AF(\text{volt}_1)} = \left(\frac{\text{volt}_1}{\text{volt}_u}\right)^{\beta_1} T(\text{volt}_u)$$
$$AF(\text{volt}_1) = \frac{T(\text{volt}_2)}{T(\text{volt}_2)}\left(\frac{\text{volt}_u}{\text{volt}_1}\right)^{\beta_1}$$
$$\boxed{AF(\text{volt}_1) = \left(\frac{\text{volt}_2}{\text{volt}_1}\right)^{\beta_1}}$$

6. Practical application of accelerated life testing methods requires some knowledge of the underlying physics/chemistry of failure. Explain why.

Knowledge of the underlying chemistry/physics allows one to appropriately choose candidate models, for example between linear and nonlinear degradation. In addition, when planning the tests, one should know something about how high is too high for acceleration factor levels before other modes are introduced and the data is irrelevant.

7. In general, planning values are needed to do test planning and to determine the sample size needed to provide a specified degree of precision.

(a) Explain why such planning values are needed.

Planning values are needed to give some idea of the expected parameter ranges. Without such information to "back out" a test plan from expected results, one might as well throw darts to pick a sample size. One needs some information to build on, and planning values is our information.

(b) Product or reliability engineers may be able to provide some useful information, but they cannot be expected to provide anything like exact parameter values (otherwise they would have no reason to run the test!). What can be done to protect against the use of potentially misspecified planning values?

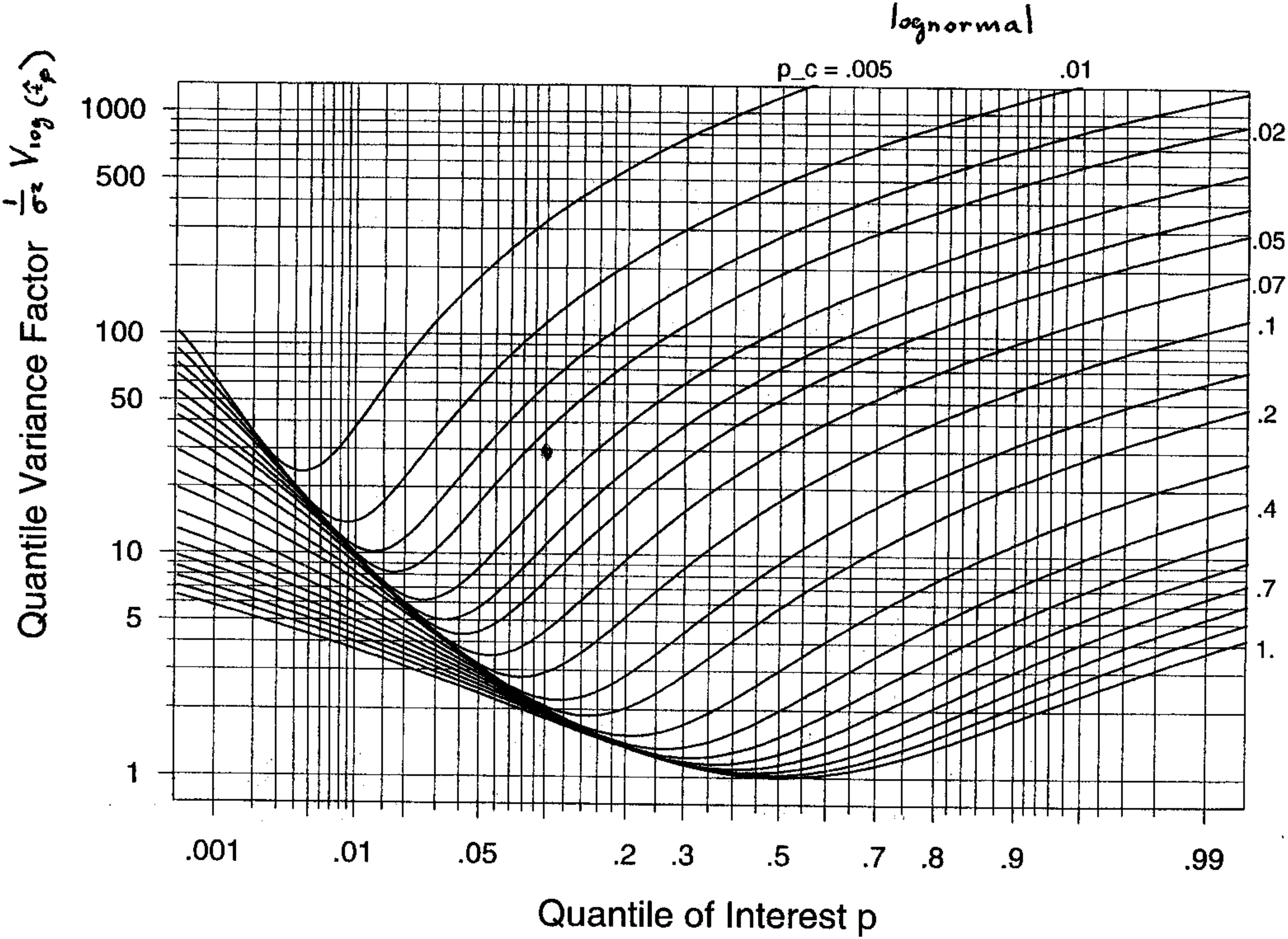
Sensitivity analysis is the best defense here. One can calculate several test plans with slightly perturbed planning values (or distributions if that is in doubt) and see how much the test plan changes. One can also use simulations to see how robust the test plans are.

8. Explain how the hazard function $h(t)$ for a continuous random variable (e.g., from a Weibull distribution) is related to a probability.

$h(t)$ itself is not a probability.
However $\Delta t\, h(t)$ is approximately the probability of failure in the next interval of time Δt given that the unit has survived until time t.

Table C.20: Normal distribution Fisher information, large-sample approximate variance-covariance matrix entries, and other factors for planning normal/lognormal distribution life tests with censored data.

ζ_c	$100\Phi_{\mathrm{nor}}(\zeta_c)$	f_{11}	f_{22}	f_{12}	$\frac{1}{\sigma^2}\mathrm{V}_{\hat\mu}$	$\frac{1}{\sigma^2}\mathrm{V}_{\hat\sigma}$	$\frac{1}{\sigma^2}\mathrm{V}_{(\hat\mu,\hat\sigma)}$	$\rho_{(\hat\mu,\hat\sigma)}$	$\frac{1}{\sigma^2}\mathrm{V}_{\hat\mu\vert\sigma}$	$\frac{1}{\sigma^2}\mathrm{V}_{\hat\sigma\vert\mu}$	ζ_c
−3.0	.13	.01467	.13583	−.04438	6001.31	647.931	1960.68	.99430	68.1891	7.36202	−3.0
−2.8	.26	.02478	.20153	−.07015	2751.23	338.313	957.667	.99264	40.3532	4.96214	−2.8
−2.6	.47	.04016	.28463	−.10589	1297.27	183.052	482.607	.99036	24.8990	3.51339	−2.6
−2.4	.82	.06245	.38264	−.15260	628.580	102.590	250.686	.98718	16.0128	2.61345	−2.4
−2.2	1.39	.09322	.48976	−.20998	312.728	59.5263	134.078	.98270	10.7269	2.04181	−2.2
−2.0	2.28	.13371	.59734	−.27592	159.661	35.7402	73.7498	.97630	7.47860	1.67408	−2.0
−1.8	3.59	.18451	.69536	−.34639	83.6383	22.1926	41.6638	.96706	5.41988	1.43811	−1.8
−1.6	5.48	.24529	.77473	−.41570	44.9858	14.2432	24.1386	.95361	4.07682	1.29078	−1.6
−1.4	8.08	.31476	.82978	−.47734	24.8920	9.44231	14.3192	.93401	3.17699	1.20513	−1.4
−1.2	11.51	.39070	.86008	−.52495	14.2242	6.46160	8.68176	.90557	2.55948	1.16269	−1.2
−1.0	15.87	.47022	.87084	−.55353	8.44766	4.56136	5.36957	.86502	2.12668	1.14831	−1.0
−.8	21.19	.55009	.87193	−.56028	5.26120	3.31921	3.38069	.80899	1.81789	1.14688	−.8
−.6	27.43	.62719	.87550	−.54498	3.47293	2.48793	2.16185	.73546	1.59442	1.14221	−.6
−.4	34.46	.69881	.89314	−.50996	2.45318	1.91942	1.40071	.64550	1.43100	1.11964	−.4
−.2	42.07	.76293	.93338	−.45948	1.86310	1.52288	.91716	.54450	1.31073	1.07138	−.2
.0	50.00	.81831	1.00000	−.39894	1.51709	1.24145	.60523	.44101	1.22203	1.00000	.0
.2	57.93	.86449	1.09172	−.33400	1.31180	1.03877	.40133	.34380	1.15675	.91599	.2
.4	65.54	.90170	1.20294	−.26976	1.18876	.89108	.26658	.25901	1.10901	.83130	.4
.6	72.57	.93069	1.32534	−.21026	1.11442	.78257	.17680	.18932	1.07447	.75452	.6
.8	78.81	.95252	1.44973	−.15819	1.06923	.70251	.11667	.13462	1.04985	.68978	.8
1.0	84.13	.96841	1.56779	−.11490	1.04168	.64344	.07634	.09325	1.03262	.63784	1.0
1.2	88.49	.97961	1.67317	−.08058	1.02488	.60004	.04936	.06294	1.02082	.59767	1.2
1.4	91.92	.98723	1.76212	−.05455	1.01467	.56847	.03141	.04136	1.01294	.56750	1.4
1.6	94.52	.99225	1.83336	−.03565	1.00852	.54583	.01961	.02643	1.00782	.54545	1.6
1.8	96.41	.99544	1.88766	−.02249	1.00485	.52990	.01197	.01641	1.00458	.52976	1.8
2.0	97.72	.99740	1.92712	−.01369	1.00270	.51896	.00712	.00987	1.00261	.51891	2.0
2.2	98.61	.99857	1.95450	−.00804	1.00147	.51166	.00412	.00576	1.00144	.51164	2.2
2.4	99.18	.99923	1.97267	−.00456	1.00078	.50693	.00231	.00325	1.00077	.50693	2.4
2.6	99.53	.99960	1.98420	−.00249	1.00040	.50398	.00126	.00177	1.00040	.50398	2.6
2.8	99.74	.99980	1.99121	−.00131	1.00020	.50221	.00066	.00093	1.00020	.50221	2.8
3.0	99.87	.99990	1.99530	−.00067	1.00010	.50118	.00033	.00047	1.00010	.50118	3.0
∞	100.00	1.00000	2.00000	.00000	1.00000	.50000	−.00000	.00000	1.00000	.50000	∞


lognormal
p_c = .005
.01
.02
.05
.07
.1
.2
.4
.7
1.
Quantile Variance Factor $\frac{1}{\sigma^2} V_{log}(\hat{t}_p)$
1000
500
100
50
10
5
1
.001
.01
.05
.2
.3
.5
.7
.8
.9
.99
Quantile of Interest p