Statistics 533  
Spring 2012  

Midterm Exam  

Name  

Key  

Instructions  

When asked to explain something, provide an explanation that could be understood by someone who does not have formal training in statistical methods. Your explanations should be clear, but concise.  

You must show all of your work  

Students may use one sheet (8.5 by 11 inches, both sides) of paper containing equations or notes.  

Students may have up to 110 minutes to complete the exam.  

There are 21 questions or question parts. Students are allowed to mark one question part as “DO NOT GRADE” without penalty.  

As soon as possible after the exam is completed, it should be scanned and emailed to wqmeeker@iastate.edu or faxed to my attention at 515-294-4040. In the case of a fax, please send email to indicate that the fax has been sent.
1. Censored data often arise in reliability and biomedical applications where data must be analyzed before all units have failed. We have learned about parametric and nonparametric methods of estimation that will accommodate right censoring. What important assumption is needed for the analysis of censored data to be correct (i.e., statistically consistent)?

   We must make an assumption of non-informative censoring. In other words, the time of censoring should provide no information about the time to which the unit would have survived if it had not been censored.

2. Provide a useful definition of the technical term “Reliability,” as it relates to engineering applications. Write one or two sentences of explanation to justify your definition.

   Various possibilities. See chapter 1 of ME1998.

3. Engineers sometimes use ordinary least squares (OLS) to fit a line through the points on a probability plot (indeed, several popular commercial packages have had this as the default method for estimating Weibull parameters). What assumptions of OLS are violated by doing this?

   1. Independent residuals
   2. Constant variance of the residuals

4. It has been suggested that the Weibull distribution should be useful for describing the time to first failure of a series system with many components, none of which dominate in limiting the reliability of the system. A series system fails when the first component fails. Why might one expect the Weibull distribution to provide a good model in this situation?

   The Weibull distribution is one of the distributions of minima.

5. Minimum sample-size demonstration tests have been popular in certain companies to demonstrate that a product has the desired reliability. Minimum sample size tests have appeal for cost reasons. Explain the disadvantage of using such tests and what can be done to improve the properties of a needed demonstration test.

   Minimum sample size tests tend to have a low probability of successful demonstration unless the actual reliability is much larger than the level to be demonstrated.
6. The delta method is widely used in statistics to estimate standard errors of nonlinear functions of parameters. The Wald method of computing confidence intervals is computationally simple, but depends on a good choice of transformation. For the one parameter exponential distribution with cdf

\[
\Pr(T \leq t) = F(t) = 1 - \left( -\frac{t}{\theta} \right), \quad t > 0,
\]

statistical theory predicts that, in small samples, the log likelihood will be approximately symmetric for the transformation $\theta^{1/3}$. Suppose that you know $\text{Var}(\hat{\theta})$ but that you need an expression for $\text{Var}(\hat{\theta}^{1/3})$ in order to compute an accurate Wald-based confidence interval for $\theta$.

(a) Give an intuitive explanation for why it is that the Wald-based interval will provide a good procedure when based on $\theta^{1/3}$.

The Wald-based interval is based on a quadratic approximation to the log likelihood. A transformation that makes the likelihood more symmetric will improve this approximation.

(b) Give a delta-method expression that can be used to compute an estimate of $\text{Var}(\hat{\theta}^{1/3})$ as a function of an estimate of $\text{Var}(\hat{\theta})$.

\[
\text{Var}(\hat{\theta}) = \left( \frac{d\theta}{d\hat{\theta}} \right)^2 \text{Var}(\hat{\theta}) \quad \text{and} \quad \text{Var}(\hat{\theta}^{1/3}) = \frac{1}{3} \theta^{-2/3} \text{Var}(\hat{\theta})
\]

$$ SE(\hat{\theta}^{1/3}) = \frac{1}{3} \hat{\theta}^{-2/3} \text{Var}(\hat{\theta})$$

(c) Explain why the delta-method approximation tends to work better (i.e., is more accurate) in large samples. Draw a picture to help you explain.

As $n$ gets larger, the distribution shrinks down and translation is needed on the closer-to-linear part of the curve.

(d) Show how you would use an estimate $\text{Var}(\hat{\theta}^{1/3})$ to compute a Wald-based approximate confidence interval for $\theta$.

compute \[ \hat{\theta}^{1/3} \pm Z_{(1-d/2)} SE(\hat{\theta}^{1/3}) = \left[ \hat{\theta}^{1/3} \pm \frac{1}{\sqrt{3}} \right] \]

Then \[ \left[ \hat{\theta}^{1/3}, \hat{\theta}^{1/3} \right] = \left[ \left( \hat{\theta}^{1/3} \right)^3, \left( \hat{\theta}^{1/3} \right)^3 \right] \]

(e) What is the distributional basis for the Wald-based approximate confidence interval based on the transformation $\theta^{1/3}$? That is, what random variable is being assumed to approximately follow a NOR(0,1) distribution.

\[
\frac{\hat{\theta}^{1/3} - \theta^{1/3}}{SE(\hat{\theta}^{1/3})} \sim N(0,1)
\]
7. There is a close relationship between the Fisher information matrix and the Observed information matrix.

(a) Give expressions for each and explain the difference between these two matrices.

\[
\text{FIM} = \mathbb{E} \left[ - \frac{\partial^2 \ell}{\partial \theta^2} \right] \quad \text{OIM} = -\frac{\partial^2 \ell}{\partial \theta^2} \bigg|_{\hat{\theta}}
\]

FIM evaluates curvature in log likelihood at the ML estimate. OIM computes the expected curvature.

(b) Explain why both are important to us in reliability applications (i.e., we do we need both of them, depending on what we are trying to do).

The OIM is easy to compute from data.

The FIM depends on the censoring distribution or the censoring mechanism, which may not be known. When planning an experiment, however, we have no data, but we can specify the censoring mechanism, so the FIM is used.

8. During one day's production of an electronic system, 500 units were manufactured. Unfortunately, all of these units were shipped without a special protective circuit that would protect the system from electrical surges. When the system is operated in a particular state and a surge occurs, the system will fail. In the first three months of operation, five failures have been reported at 23, 43, 50, 78, and 91 days. Suppose that all units were put into service at about the same time, that the exponential distribution provides a reasonable model for the time to failure of the units in the field, and that the reported failure times were the times between installation and the time when the data were analyzed.

(a) Compute the maximum likelihood estimate of the mean time to failure, assuming that the "exact failure" density approximation is adequate (as it is in this case).

\[
\text{MTTF} = \hat{\theta} = \frac{12 + 43 + 50 + 78 + 91 + 495(91)}{5} = 9066 \text{ days}
\]

Assuming 91 days in first three months.

(b) Compute an approximate 95% confidence interval for the exponential mean time to failure.

There are several possibilities. Wald is the easiest.

\[
\hat{\theta} \pm 1.96 \times \frac{\hat{\theta}}{\sqrt{\hat{\theta}}}
\]

\[
9066 \pm 1.96 \times 4.0544 = [1119.4, 1701.6]
\]

(c) Compute an approximate 95% confidence interval for the fraction failing after one year, based on the exponential distribution.

\[
\hat{F}(t) = 1 - \exp \left[ -\frac{t}{\hat{\theta}} \right] = 1 - \exp \left[ -\frac{365}{9066} \right] = 0.0395
\]

Similarly,

\[
\left[ \hat{E}(365), \hat{F}(365) \right] = \left[ \frac{0.0212}{0.0395}, 0.2782 \right]
\]
9. The Weibull distribution is a popular model that is frequently used to describe a time to failure distribution. The Weibull cdf is \( F(t) = 1 - \exp[-(t/\eta)^\beta] \). Suppose that you have 10 ordered observations \( t_1, \ldots, t_{10} \) from a Weibull distribution.

(a) Explain how you compute a nonparametric estimate of \( F(t) \) based upon the available data. Because there is no censoring, the Kaplan-Meier estimate is the same as the empirical distribution function, jumping from \( (i-1)/n \) to \( i/n \) at failure \( t_i \), \( i=1, \ldots, n \). To get an estimate of \( F(t) \) at \( t \), one might use \( (i-\frac{1}{2})/n \), half the jump height.

(b) Derive an expression for the Weibull quantile function.

\[
-t_{1-p} = \left( \frac{t}{\eta} \right)^\beta \quad \text{solve for } t_{1-p} \]

\[
-t_{1-p} = \frac{-\log(1-p)}{\beta} = \frac{\log(1-p)}{\beta}
\]

so \( t_{1-p} = \eta \left[ -\log(1-p) \right]^{1/\beta} \)

(c) Suppose that you need to make a Weibull plot. You have only have paper, pencil, and a calculator. How would you transform the nonparametric estimate of \( F(t) \) so that plotting on linear axes would provide a valid Weibull plot of the data?

from (b) \( \log(t_{1-p}) = \log(\beta) + \frac{1}{\beta} \log[-\log(1-p)] \)

Thus one should plot \( \log(t_{(i)}) \) versus \( \log[-\log(1-p_i)] \)

where \( p_i = \frac{i-\frac{1}{2}}{n} \) from (a).

10. A computer hard disk manufacturer tests a large number of disk drives for some number of hours to provide some assurance that no defective units will be put into service. This testing time is part of a "burn-in" process to reduce the number of defective units that are shipped. Suppose that for a new high-reliability model, over the past three months, 1000 units have been tested for 24 hours each and no failures were found.

\( n=1000 \)
\( t_e=24 \)
\( \therefore \) at 5 years \( t_f = 24(5 \times 365) = 43824 \) hours

(a) What quantitative statement can be made about disk drive reliability at five years if the Weibull shape parameter \( \beta \) is assumed to be equal to one?

\[
\eta = \left[ \frac{2 \sum t_i^B}{\sum (i-\alpha)^2} \right]^{1/B} = \left[ \frac{\sum 1000 \cdot 24^B}{59915} \right]^{1/1} = 8011.3 \text{ hours. So } S = \exp \left[ -\frac{43824}{8011.3} \right] = 0.00421
\]

Thus we are 95\% confident that reliability \( > .00421 \).

(b) What quantitative statement can be made about disk drive reliability at five years if the Weibull shape parameter \( \beta \) is assumed to be equal to two?

\[
\eta = \left[ \frac{\sum 1000 \cdot 24^{2B}}{59915} \right]^{1/2} = 438.5 \text{ hours so } S_{(1826)} = \exp \left[ -\frac{43824}{8011.3} \right] = 0.997
\]

(c) What quantitative statement can be made about disk drive reliability if no assumption can be made about the underlying time to failure distribution?

Even if \( \beta \) is unknown, we can still make a binomial-based nonparametric statement about \( S(24) \) at 24 hours.

\[
S(24) = \left[ 1 + \frac{1000}{\binom{1826}{24} \cdot 2000} \right]^{-24} = 0.003 \text{ so } S(24) = 0.997
\]