Module 11

Transfer Function Models

Class notes for Statistics 451: Applied Time Series
Iowa State University
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Segment 1
Introduction to Transfer Function Time Series Models

Module II

Transfer Function Models
Notes on the Transfer Function Model

Transfer Function (Dynamic Regression) Model

Dynamic Regression Model for Effect of Advertising on Sales

Transfer Function to Realization Filter

Innovation to Realization Filter

Gas Furnace Percent CO in Outlet Gas
Some Dynamic Regression Models

Univariate Models for the Gas Furnace Input Rate,

Univariate AR(5) Model for Gas Furnace Percent CO

Residuals

Univariate AR(9):

Regression on present and past values of input

Univariate AR(4) Model for Gas Furnace Input Rate,

\[ y_t = \theta x_t + \phi y_{t-1} + \epsilon_t \]

\[ 95\% \text{ Prediction Interval} \]

Coded Gas Rate (input)

For Percent CO\text{2} in Outlet Gas

Part 1

Part 2

Module II
The Cross-Correlation Function and an Introduction to Transfer Function Time Series Model Identification

Cross Correlation Function

The cross covariance function $\gamma_{xy}(k) = E[(x_t - \mu_x)(y_{t+k} - \mu_y)]$ describes, for $k > 0$, the relationship between future values of $y$ and current values of $x$. The cross correlation function $\rho_{xy}(k) = \gamma_{xy}(k)/\sigma_x \sigma_y$ gives the correlation between $x_t$ and $y_{t+k}$ (or, equivalently, between $x_t-k$ and $y_t$).

To see how present values of $x$ might be related to past values of $y$, use $\rho_{xy}(-k) = \rho_{yx}(k)$.

Notes on Cross Correlation Function

• Both $x_t$ and $y_t$ must be stationary for cross correlations to be defined.

• To judge whether a sample CCF value is significantly different from 0, use

$$t = \frac{\hat{\rho}_{xy}(k)}{S(\hat{\rho}_{xy}(k))} \approx \sqrt{n}$$

where

$$S(\hat{\rho}_{xy}(k)) = \sqrt{\frac{1}{n-k}}$$

Compare with standard normal quantiles (signal if outside $\pm 2$).

• As with ACF and PACF, commonly used tests and standard errors for the CCF are only approximate.

• The sample CCF is difficult to interpret unless either $x_t$ or $y_t$ is white noise.

Data for Computing Sample Cross Correlations Between $x_t$ and $y_{t+k}$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\hat{\rho}_{xy}(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.348</td>
</tr>
<tr>
<td>2</td>
<td>0.339</td>
</tr>
<tr>
<td>3</td>
<td>0.373</td>
</tr>
<tr>
<td>4</td>
<td>0.441</td>
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<td>5</td>
<td>0.461</td>
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<td>6</td>
<td>0.441</td>
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<tr>
<td>7</td>
<td>0.348</td>
</tr>
<tr>
<td>8</td>
<td>0.339</td>
</tr>
<tr>
<td>9</td>
<td>0.373</td>
</tr>
</tbody>
</table>

Cross Correlation Function (CCF)

The sample cross correlation function $\hat{\rho}_{xy}(k) = \hat{\gamma}_{xy}(k)/\hat{\sigma}_x \hat{\sigma}_y$ gives the sample correlation between $x_t$ and $y_{t+k}$ (or, equivalently, between $x_t-k$ and $y_t$).

To see how $x_t$ is correlated with past values of $y$, use $\hat{\rho}_{xy}(-k)$.

Sample Cross Correlation Function (CCF)

The cross correlation function of $y$ and current values of $x$ of $\rho_{xy}(k)$ gives the correlation between $x_t$ and $y_{t+k}$ (or, equivalently). The cross covariance function $\gamma_{xy}(k)$ of $y$ and future values of $x$ for $k < 0$, the relationship between future values of $y$ and current values of $x$. The cross covariance function $\gamma_{xy}(k)$ is $E[(x_t - \mu_x)(y_{t+k} - \mu_y)]$.

Cross Correlation Function Between Gas Furnace Input Rate and Percent CO\(_2\) in Outlet Gas

Cross Correlation Lag

CCF

Between gasrx.d and gasry.d with model null.model

The Cross-Correlation Function and an Introduction to Transfer Function Time Series Model Identification

Segment 3

Module II
Prewhitening for Transfer Function Time Series Model

Cross Correlation Function Between Prewhitened Gas

Cross Correlation Function Between Prewhitened Gas

The transfer function model can be written as

\[
\frac{\beta(B)y_t}{\pi(B)x_t} = \nu_t(B)\gamma_t + \pi_t x_t
\]

which is a white-noise input process, allowing easy identification of the model by taking the cross correlation function as

\[
E(\pi_t x_{t+k}) = \sigma^2 \delta_{tk}
\]

where the cross correlations between the residuals and the input are zero, giving the cross correlation function of the model as

\[
E(\pi_t x_{t+k}) = \sigma^2 \delta_{tk}
\]

and

\[
E(y_{t+k} x_t) = \sigma^2 \delta_{tk}
\]

for the model. As an example, the cross correlation function of the residuals of the model is obtained by subtracting \(\nu_t(B)\gamma_t\) from the original series, giving

\[
E(y_{t+k} - \nu_t(B)\gamma_t x_t) = \sigma^2 \delta_{tk}
\]

The cross correlation function of the residuals is given by

\[
E(y_{t+k} - \nu_t(B)\gamma_t x_t) = \sigma^2 \delta_{tk}
\]

for the model. As an example, the cross correlation function of the residuals of the model is obtained by subtracting \(\nu_t(B)\gamma_t\) from the original series, giving

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E(y_{t+k} - \nu_t(B)\gamma_t x_t) = \sigma^2 \delta_{tk}
\]

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\]

for the model. As an example, the cross correlation function of the residuals of the model is obtained by subtracting \(\nu_t(B)\gamma_t\) from the original series, giving

\[
E(y_{t+k} - \nu_t(B)\gamma_t x_t) = \sigma^2 \delta_{tk}
\]
This model has 10 parameters (plus \(\phi\)).

\[
1 + 12\phi_1 - 1\phi_2 - 1\phi_3 + 4\phi_4 - 3\phi_5 + \cdots + 1 - 1\phi_5 = 0
\]

\[
1 + 12\phi_1 - 1\phi_2 - \cdots - 1\phi_5 = 0
\]

\[
h_0 + 1z(\phi_1 + \cdots + \phi_5) - 1 = \mu_0(\phi_1 + \cdots + \phi_5 - 1)
\]

\[
h_0 + 1z(\phi_1 + \cdots + \phi_5) = \mu_0(\phi_1 + \cdots + \phi_5)
\]

\[
\mu_0 + 1z(\phi_1 + \cdots + \phi_5) = \mu
\]
Parsimonious Transfer Function Model for Gas Furnace Percent CO in Outlet Gas

- This model has 7 parameters (thus φ).
- Typically 3 or 4.

Forecasting Transfer Function Output with Stochastic Variables by Using New \( u_t \) to Help Explain the Residuals from the Model, Making a Multiple Input Transfer Function Model and New Variants of the Current Model.

Fit and check the tentative dynamic regression model.

Identity tentative dynamic regression model from CCF.

Use the \( u_t \) model to "filter" both \( x_t \) and \( y_t \), examining the model \( u_t \) alone as a baseline for comparison.

Strategy for Identifying a Transfer Function Model

For a multiple input transfer function model, add new variables.

- \( \frac{1}{(1 - B)^3} \sum_1^3 \frac{\phi_1}{\delta_1} + \sum_1^3 \frac{\phi_2}{\delta_2} + \frac{\phi_3}{\delta_3} \phi_3 \) represents the \( \phi \) parameters.

Forecasting Transfer Function Output with Stochastic (so that \( u_t \) is stochastic).