Module 12

Transfer Function and Intervention Analysis Case Studies and an Introduction to Multivariate (Vector) Time Series Models

Class notes for Statistics 451: Applied Time Series
Iowa State University

Copyright 2016 W. Q. Meeker.

May 7, 2016
5h 40min
Module 12

Segment 1

Crest Market Share Intervention Model
Tooth Paste Market

- During the 1950s, Colgate was the toothpaste market leader with about 40% of the market.
- Proctor and Gamble invested heavily in the development of a new “therapeutic” toothpaste and 1954 introduced their new brand Crest that used a fluoride-containing compound as a component to help prevent tooth decay.
- Over the next six years Crest maintained a reasonably steady 15% of the market.
- Proctor and Gamble conducted extensive experiments to compare Crest with toothpaste that did not contain the fluoride-containing compound.
- In August 1960, they were able to convince the American Dental Association to make an endorsement saying that Crest was “an effective anti-caries” toothpaste.
- Crest’s market share immediately increased dramatically and within two years, they were the market leader.
Colgate Market Share Weekly Data

Colgate Market Share

Percent

Year


12 - 4
Crest Market Share Weekly Data

Crest Market Share

<table>
<thead>
<tr>
<th>Year</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1958</td>
<td>0.1</td>
</tr>
<tr>
<td>1959</td>
<td>0.2</td>
</tr>
<tr>
<td>1960</td>
<td>0.3</td>
</tr>
<tr>
<td>1961</td>
<td>0.4</td>
</tr>
<tr>
<td>1962</td>
<td>0.5</td>
</tr>
<tr>
<td>1963</td>
<td></td>
</tr>
</tbody>
</table>
Crest Market Share iden Output No Differencing

Crest Market Share

w = Percent

Range−Mean Plot

Time

ACF

PACF
Crest Market Share iden Output One Difference

Crest Market Share
\[ w = (1-B^1) \text{ Percent} \]
Crest Market Share esti Output Part 1
IMA(1,1) (Model 1)

Residuals versus Time

Residual ACF

Residuals versus Fitted Values

Residual ACF
Crest Market Share esti Output Part 2
IMA(1,1) (Model 1)
Models Fit to the Crest Market Share Data

- Model 1: IMA(1,1)
  
  \[ y_t = \frac{(1 - \theta_1 B)}{(1 - B)} a_t \]

- Model 2: ARMA(1,1)
  
  \[ y_t = \frac{(1 - \theta_1 B)}{(1 - \phi_1 B)} a_t \]

- Model 3: IMA(1,1) with intervention term
  
  \[ y_t = \frac{\omega_0}{(1 - B)} I_t + \frac{(1 - \theta_1 B)}{(1 - B)} a_t \]

- Model 4: ARMA(1,1) with intervention term
  
  \[ y_t = \frac{\omega_0}{(1 - B)} I_t + \frac{(1 - \theta_1 B)}{(1 - \phi_1 B)} a_t \]

where

\[ I_t = \begin{cases} 
0, & \text{before August 1960}, \\
1, & \text{after August 1960}, 
\end{cases} \]
## Comparison of Models for the Crest Market Share Data

<table>
<thead>
<tr>
<th>Model</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Noise ARIMA$(p, d, q)$</td>
<td>(1,1,0)</td>
<td>(1,0,1)</td>
<td>(1,1,0)</td>
<td>(1,0,1)</td>
</tr>
<tr>
<td>Intervention Term</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>—</td>
<td>0.992</td>
<td>—</td>
<td>0.971</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(151.2)</td>
<td></td>
<td>(36.3)</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>0.664</td>
<td>0.658</td>
<td>0.772</td>
<td>0.749</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(15.4)</td>
<td>(14.8)</td>
<td>(16.9)</td>
</tr>
<tr>
<td>$\theta_0$</td>
<td>—</td>
<td>0.269</td>
<td>—</td>
<td>0.167</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2.9)</td>
<td></td>
<td>(5.5)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>—</td>
<td>—</td>
<td>0.151</td>
<td>0.179</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(5.2)</td>
</tr>
<tr>
<td>$S$</td>
<td>0.046</td>
<td>0.045</td>
<td>0.044</td>
<td>0.044</td>
</tr>
<tr>
<td>AIC$_c$</td>
<td>$-914.0$</td>
<td>$-912.1$</td>
<td>$-931.9$</td>
<td>$-933.5$</td>
</tr>
<tr>
<td>$-2 \log(\text{Likelihood})$</td>
<td>$-917.9$</td>
<td>$-920.1$</td>
<td>$-937.9$</td>
<td>$-943.5$</td>
</tr>
<tr>
<td>Ljung-Box $\chi^2_6$</td>
<td>3.38</td>
<td>4.89</td>
<td>2.95</td>
<td>3.86</td>
</tr>
<tr>
<td>Ljung-Box p-value</td>
<td>0.64</td>
<td>0.56</td>
<td>0.81</td>
<td>0.70</td>
</tr>
</tbody>
</table>
Crest Market Share esti Output Part 1
ARMA(1,1) (Model 2)

Residuals versus Time

Residuals versus Fitted Values

Residual ACF
Crest Market Share esti Output Part 2
ARMA(1,1) (Model 2)

Crest Market Share

Normal Q–Q Plot
Crest Market Share esti Output Part 2
IMA(1,1) (Model 1)

Crest Market Share

Normal Q–Q Plot

Normal Scores

Crest Market Share esti Output Part 1
IMA(1,1) with Intervention Term (Model 3)

Residuals versus Time

Residuals versus Fitted Values

Residual ACF
Crest Market Share esti Output Part 2
IMA(1,1) with Intervention Term (Model 3)

Crest Market Share

Normal Q–Q Plot

Normal Scores

12 - 16
Crest Market Share esti Output Part 2
IMA(1,1) (Model 1)

Crest Market Share

Normal Q–Q Plot
Crest Market Share esti Output Part 1
ARMA(1,1) with Intervention Term (Model 4)
Crest Market Share esti Output Part 2
ARMA(1,1) with Intervention Term (Model 4)

Crest Market Share

Normal Q–Q Plot
Crest Market Share esti Output Part 2
ARMA(1,1) (Model 2)

Crest Market Share

Normal Q–Q Plot
Crest Market Share esti Output Part 2
IMA(1,1) with Intervention Term (Model 3)
Models Fit to the Crest Market Share Data

• Model 1: IMA(1,1)

\[ y_t = \frac{(1 - 0.664B)}{(1 - B)}a_t \]

• Model 2: ARMA(1,1)

\[ y_t = \frac{(1 - 0.658B)}{(1 - 0.992B)}a_t \]

• Model 3: IMA(1,1) with intervention term

\[ y_t = \frac{0.151}{(1 - B)}I_t + \frac{(1 - 0.772B)}{(1 - B)}a_t \]

• Model 4: ARMA(1,1) with intervention term

\[ y_t = \frac{0.179}{(1 - B)}I_t + \frac{(1 - 0.749B)}{(1 - 0.971B)}a_t \]

where

\[ I_t = \begin{cases} 
0, & \text{before August 1960}, \\
1, & \text{after August 1960}, 
\end{cases} \]
Comparison of Models for the Crest Market Share Data

<table>
<thead>
<tr>
<th>Model</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Noise ARIMA ((p, d, q))</td>
<td>((1,1,0))</td>
<td>((1,0,1))</td>
<td>((1,1,0))</td>
<td>((1,0,1))</td>
</tr>
<tr>
<td>Intervention Term</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(\phi_1)</td>
<td>—</td>
<td>0.992</td>
<td>—</td>
<td>0.971</td>
</tr>
<tr>
<td>(\theta_1)</td>
<td>0.664</td>
<td>0.658</td>
<td>0.772</td>
<td>0.749</td>
</tr>
<tr>
<td>(</td>
<td>\theta_1</td>
<td>)</td>
<td>(15.4)</td>
<td>(14.8)</td>
</tr>
<tr>
<td>(\theta_0)</td>
<td>—</td>
<td>0.269</td>
<td>—</td>
<td>0.167</td>
</tr>
<tr>
<td>(</td>
<td>\theta_0</td>
<td>)</td>
<td>(2.9)</td>
<td>(5.5)</td>
</tr>
<tr>
<td>(\beta_1)</td>
<td>—</td>
<td>—</td>
<td>0.151</td>
<td>0.179</td>
</tr>
<tr>
<td>(</td>
<td>\beta_1</td>
<td>)</td>
<td>(5.2)</td>
<td>(4.8)</td>
</tr>
<tr>
<td>(S)</td>
<td>0.046</td>
<td>0.045</td>
<td>0.044</td>
<td>0.044</td>
</tr>
<tr>
<td>(\text{AIC}_c)</td>
<td>(-914.0)</td>
<td>(-912.1)</td>
<td>(-931.9)</td>
<td>(-933.5)</td>
</tr>
<tr>
<td>(-2\log(\text{Likelihood}))</td>
<td>(-917.9)</td>
<td>(-920.1)</td>
<td>(-937.9)</td>
<td>(-943.5)</td>
</tr>
<tr>
<td>Ljung-Box (\chi^2_6)</td>
<td>3.38</td>
<td>4.89</td>
<td>2.95</td>
<td>3.86</td>
</tr>
<tr>
<td>Ljung-Box (p)-value</td>
<td>0.64</td>
<td>0.56</td>
<td>0.81</td>
<td>0.70</td>
</tr>
</tbody>
</table>
Module 12

Segment 2

Proper Modeling in Time Series Regression

Attempt to Predict the Financial Times Index
Attempt to Predict the Financial Times Index

- Two economists (Coen and Gomme) and one well-known statistician (Kendall) published a paper in a leading statistics journal (Journal of the Royal Statistical Society) claiming to have found a predictive model for the Financial Times Index.

- The ability to use leading indicators to predict such a financial index goes against economic theory (the efficient market hypothesis)

- The quarterly Financial Times Index data started with Q2 1954 and ran to the end of 1966 (51 quarters).

- The OLS regression model for Financial Times Index with the Financial Times Commodity Index (lagged 7 quarters) and UK automobile production (lagged 6 quarters) as explanatory variables indicated extremely strong statistical significance.
Financial Times Index

Year

Index

Financial Times Index and Proposed Leading Indicators

**Financial Times Index**

- Index: 150, 200, 250, 300, 350

**Financial Times Commodity Index (lagged 7 quarters)**

- Index: 80, 85, 90, 95

**UK Automobile Production (lagged 6 quarters)**

- Thousands: 150, 200, 250, 300, 350
Ordinary Least Squares (OLS) Regression
Relating Financial Times Index
and Proposed Leading Indicators

\[ \text{lm(formula = FTindex \sim FTCindex + UKCarProd)} \]

\[
\text{FTindex} = \beta_0 + \beta_1 \text{FTCindex} + \beta_2 \text{UKCarProd} + a_t
\]

OLS assumes **independent** residuals.

Coefficients:

|                | Estimate | Std. Error | t value | Pr(>|t|)   |
|----------------|----------|------------|---------|------------|
| (Intercept)    | 653.15969| 56.53231   | 11.554  | 1.82e-15 ***|
| FTCindex       | -6.12736 | 0.61987    | -9.885  | 3.70e-13 ***|
| UKCarProd      | 0.47468  | 0.03363    | 14.114  | < 2e-16 *** |

Residual standard error: 22.29 on 48 degrees of freedom
Multiple R-squared: 0.9018, Adjusted R-squared: 0.8977
Residual ACF for Ordinary Least Squares Regression
Relating Financial Times Index
and Proposed Leading Indicators

acf(residuals(FTindexReg.out))
Regression Relating Financial Times Index and Proposed Leading Indicators

White Noise Model for the Residuals

Model 6 Part 1 esti Output

Financial Times Index

Residuals versus Time
ARIMA(0,0,0) on w= Index  regr variables= rbind(FTindexXmat, FTindexXmatXnew)[, c(2, 3)]

Residuals versus Fitted Values

Residual ACF
Regression Relating Financial Times Index and Proposed Leading Indicators
White Noise Model for the Residuals
Model 6 esti Tabular Output

ARIMA(0,0,0)
ARIMA estimation results:
AICc: 466.3
S: 21.63

Parameter Estimation Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MLE</th>
<th>se</th>
<th>t.ratio</th>
<th>95% lower</th>
<th>95% upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>intercept</td>
<td>653.160</td>
<td>54.8445</td>
<td>11.9</td>
<td>545.665</td>
<td>760.655</td>
</tr>
<tr>
<td>FTCindex</td>
<td>-6.127</td>
<td>0.6014</td>
<td>-10.2</td>
<td>-7.306</td>
<td>-4.949</td>
</tr>
<tr>
<td>UKCarProd</td>
<td>0.475</td>
<td>0.0326</td>
<td>14.5</td>
<td>0.411</td>
<td>0.539</td>
</tr>
</tbody>
</table>
Regression Relating Financial Times Index and Proposed Leading Indicators
White Noise Model for the Residuals
Model 6 Part 2 esti Output

Actual Values, Fitted Values and Predictions with 95% Prediction Intervals
Financial Times Index
ARIMA(0,0,0) on w= Index regr variables= rbind(FTindexXmat, FTindexXmatXnew), c(2, 3)

Normal Q–Q Plot
Models Fit to the Financial Times Index Data

- Model 6: Regression with ARIMA(0,0,0) (independent) errors
  \[ y_t = \beta_0 + \beta_1 FTCindex_{t-7} + \beta_2 UKCarProd_{t-6} + a_t \]

- Model 8: Regression with ARIMA(0,1,0) (random walk) errors
  \[ y_t = \beta_1 FTCindex_{t-7} + \beta_2 UKCarProd_{t-6} + \frac{a_t}{(1-B)} \]

- Model 4: ARIMA(0,1,0) (random walk) errors
  \[ y_t = \frac{a_t}{(1-B)} = y_{t-1} + a_t \]

- Model 2: IMA(1,1) errors
  \[ y_t = \frac{(1-\theta_1B)}{(1-B)}a_t = y_{t-1} - \theta_1a_{t-1} + a_t \]
# Comparison of Models for the Financial Times Index

<table>
<thead>
<tr>
<th>Model Number</th>
<th>6</th>
<th>8</th>
<th>4</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(0,0,0)</td>
<td>(0,1,0)</td>
<td>(0,1,0)</td>
<td>(0,1,1)</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( \beta_0 )</td>
<td>653.2</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>(11.9)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>-6.13</td>
<td>-1.36</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>(10.18)</td>
<td>(1.20)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>0.47</td>
<td>0.183</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>(14.5)</td>
<td>(2.31)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sig. ( \hat{\rho}_k(\hat{a}) )</td>
<td>1</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( S )</td>
<td>21.6</td>
<td>17.2</td>
<td>18.33</td>
<td>18.13</td>
</tr>
<tr>
<td>AIC(_c)</td>
<td>466.3</td>
<td>433.0</td>
<td>434.8</td>
<td>435.7</td>
</tr>
<tr>
<td>(-2 \log(\text{Likelihood}))</td>
<td>458.3</td>
<td>427.0</td>
<td>432.8</td>
<td>431.7</td>
</tr>
<tr>
<td>Ljung-Box ( \chi^2_6 )</td>
<td>19.8</td>
<td>4.37</td>
<td>1.08</td>
<td>1.46</td>
</tr>
<tr>
<td>Ljung-Box ( \chi^2_6 ) p-value</td>
<td>0.003</td>
<td>0.63</td>
<td>0.98</td>
<td>0.96</td>
</tr>
</tbody>
</table>
Regression Relating Financial Times Index and Proposed Leading Indicators
Random Walk Model for the Residuals
Model 8 Part 1 esti Output

Financial Times Index
Residuals versus Time
ARIMA(0,1,0) on w= Index  regr variables= rbind(FTindexXmat[, c(2, 3)], FTindexXmatXnew[, c(2, 3)])

Residuals versus Fitted Values

Residual ACF
Regression Relating Financial Times Index and Proposed Leading Indicators
Random Walk Model for the Residuals
Model 8 esti Tabular Output

ARIMA(0,1,0)
ARIMA estimation results:
AICc: 433
S: 17.3

Parameter Estimation Results

<table>
<thead>
<tr>
<th></th>
<th>MLE</th>
<th>se</th>
<th>t.ratio</th>
<th>95% lower</th>
<th>95% upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>FTCindex</td>
<td>-1.360</td>
<td>1.137</td>
<td>-1.20</td>
<td>-3.5884</td>
<td>0.868</td>
</tr>
<tr>
<td>UKCarProd</td>
<td>0.183</td>
<td>0.079</td>
<td>2.31</td>
<td>0.0277</td>
<td>0.337</td>
</tr>
</tbody>
</table>
Regression Relating Financial Times Index and Proposed Leading Indicators
Random Walk Model for the Residuals
Model 8 Part 2 esti Output

Actual Values, Fitted Values and Predictions with 95% Prediction Intervals

Financial Times Index

ARIMA(0,1,0) on w= Index  regr variables= rbind(FTindexXmat[, c(2, 3)], FTindexXmatXnew[, c(2, 3)])
Regression Relating Financial Times Index and Proposed Leading Indicators
White Noise Model for the Residuals
Model 6 Part 2 esti Output

Actual Values, Fitted Values and Predictions with 95% Prediction Intervals
Financial Times Index
ARIMA(0,0,0) on w= Index regr variables= rbind(FTindexXmat, FTindexXmatXnew)[, c(2, 3)]
Financial Times Index
Ignoring Leading Indicators
Random Walk Model
Model 4 estimation Tabular Output

Estimation/Forecasting Output for Financial Time Index
ARIMA(0,1,0)
ARIMA estimation results:
Series: FTindex.tsd
AICc: 434.8
S: 18.3
Financial Times Index
Ignoring Leading Indicators
Random Walk Model
Model 4 Part 2 esti Output

Actual Values, Fitted Values and Predictions with 95% Prediction Intervals

ARIMA(0,1,0) on w= Index

Normal Q–Q Plot
Regression Relating Financial Times Index and Proposed Leading Indicators
Random Walk Model for the Residuals
Model 8 Part 2 esti Output

Actual Values, Fitted Values and Predictions with 95% Prediction Intervals

Financial Times Index
ARIMA(0,1,0) on w= Index  regr variables= rbind(FTindexXmat[, c(2, 3)], FTindexXmatXnew[, c(2, 3)])
Models Fit to the Financial Times Index Data

- Model 6: Regression with ARIMA(0,0,0) (independent) errors
  \[ y_t = 653.2 - 6.13 \text{FTCindex}_{t-7} + 0.47 \text{UKCarProd}_{t-6} + a_t \]

- Model 8: Regression with ARIMA(0,1,0) (random walk) errors
  \[ y_t = -1.36 \text{FTCindex}_{t-7} + 0.183 \text{UKCarProd}_{t-6} + \frac{a_t}{(1 - B)} \]

- Model 4: ARIMA(0,1,0) (random walk) errors
  \[ y_t = \frac{a_t}{(1 - B)} = y_{t-1} + a_t \]

- Model 2: IMA(1,1) errors
  \[ y_t = \frac{(1 + 0.15B)}{(1 - B)}a_t = y_{t-1} + 0.15a_{t-1} + a_t \]
### Comparison of Models for the Financial Times Index

<table>
<thead>
<tr>
<th>Model Number</th>
<th>6</th>
<th>8</th>
<th>4</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0,0)</td>
<td>(0,1,0)</td>
<td>(0,1,0)</td>
<td>(0,1,1)</td>
<td></td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>(-0.15)</td>
</tr>
<tr>
<td>( \beta_0 )</td>
<td>653.2</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( (11.9) )</td>
<td>(11.9)</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>(-6.13)</td>
<td>(-1.36)</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( (\text{-10.18}) )</td>
<td>(\text{-10.18})</td>
<td>(\text{-1.20})</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>0.47</td>
<td>0.183</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( (14.5) )</td>
<td>(14.5)</td>
<td>(2.31)</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( \text{Sig. } \hat{\rho}_k(\hat{a}) )</td>
<td>1</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( S )</td>
<td>21.6</td>
<td>17.2</td>
<td>18.33</td>
<td>18.13</td>
</tr>
<tr>
<td>( AIC_c )</td>
<td>466.3</td>
<td>433.0</td>
<td>434.8</td>
<td>435.7</td>
</tr>
<tr>
<td>(-2 \log(\text{Likelihood}))</td>
<td>458.3</td>
<td>427.0</td>
<td>432.8</td>
<td>431.7</td>
</tr>
<tr>
<td>Ljung-Box ( \chi^2 )</td>
<td>19.8</td>
<td>4.37</td>
<td>1.08</td>
<td>1.46</td>
</tr>
<tr>
<td>Ljung-Box ( \chi^2 ) ( p )-value</td>
<td>0.003</td>
<td>0.63</td>
<td>0.98</td>
<td>0.96</td>
</tr>
</tbody>
</table>
Module 12

Segment 3

Using a Transfer Function/Intervention Model to Explain Bad Debt Collections
Bad Debt Collections and Leading Indicator

**Time Series BadDebtCollected**

<table>
<thead>
<tr>
<th>Time</th>
<th>BadDebtCollected</th>
</tr>
</thead>
<tbody>
<tr>
<td>1973.0</td>
<td>10</td>
</tr>
<tr>
<td>1973.5</td>
<td>20</td>
</tr>
<tr>
<td>1974.0</td>
<td>30</td>
</tr>
<tr>
<td>1974.5</td>
<td>40</td>
</tr>
<tr>
<td>1975.0</td>
<td>50</td>
</tr>
<tr>
<td>1975.5</td>
<td>60</td>
</tr>
<tr>
<td>1976.0</td>
<td>70</td>
</tr>
<tr>
<td>1976.5</td>
<td>80</td>
</tr>
</tbody>
</table>

- Mean: 56.903226
- Std: 20.432982
- N: 31
- Zero Mean ADF: -1.173664
- Single Mean ADF: 0.588564
- Trend ADF: -2.167528

**Input Time Series Panel**

**Input Series: OutstandingBadDebt**

<table>
<thead>
<tr>
<th>Time</th>
<th>OutstandingBadDebt</th>
</tr>
</thead>
<tbody>
<tr>
<td>1973.0</td>
<td>10</td>
</tr>
<tr>
<td>1973.5</td>
<td>20</td>
</tr>
<tr>
<td>1974.0</td>
<td>30</td>
</tr>
<tr>
<td>1974.5</td>
<td>40</td>
</tr>
<tr>
<td>1975.0</td>
<td>50</td>
</tr>
<tr>
<td>1975.5</td>
<td>60</td>
</tr>
<tr>
<td>1976.0</td>
<td>70</td>
</tr>
<tr>
<td>1976.5</td>
<td>80</td>
</tr>
</tbody>
</table>

- Mean: 46.604651
- Std: 21.096831
- N: 43
- Zero Mean ADF: -1.0591
- Single Mean ADF: 0.741381
- Trend ADF: -2.876609
Bad Debt Collections Variables

- **Response:**
  \( y_t \) is the amount of bad debt collected in month \( t \)

- **Leading indicator:**
  \( x_t \) is the amount of outstanding bad debt on the last day of month \( t \)

- **Intervention variables:**

  - **Anticipation:**
    \[ I_{1t} = \begin{cases} 
    1, & \text{in December 1974,} \\
    0, & \text{otherwise,} 
    \end{cases} \]

  - **Change:**
    \[ I_{2t} = \begin{cases} 
    0, & \text{before January 1975,} \\
    1, & \text{after January 1975,} 
    \end{cases} \]
Bad Debt Collections Intervention Variables

**Input Series: Anticipation**

<table>
<thead>
<tr>
<th>Time</th>
<th>Mean</th>
<th>Std</th>
<th>N</th>
<th>Zero Mean ADF</th>
<th>Single Mean ADF</th>
<th>Trend ADF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1973.0</td>
<td>0.0232558</td>
<td>0.1507149</td>
<td>43</td>
<td>-6.403124</td>
<td>-6.480741</td>
<td>-6.405667</td>
</tr>
</tbody>
</table>

**Input Series: Change**

<table>
<thead>
<tr>
<th>Time</th>
<th>Mean</th>
<th>Std</th>
<th>N</th>
<th>Zero Mean ADF</th>
<th>Single Mean ADF</th>
<th>Trend ADF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1973.0</td>
<td>0.4186047</td>
<td>0.4933303</td>
<td>43</td>
<td>0</td>
<td>0.821342</td>
<td>-2.094627</td>
</tr>
</tbody>
</table>

12 - 49
Bad Debt Collections
IMA(1,1) Model Estimates

Model: IMA(1, 1) No Intercept

Model Summary

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>DF</td>
<td>29</td>
<td>Stable</td>
<td>Yes</td>
</tr>
<tr>
<td>Sum of Squared Errors</td>
<td>2538.21973</td>
<td>Invertible</td>
<td>Yes</td>
</tr>
<tr>
<td>Variance Estimate</td>
<td>87.5248183</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>9.35546997</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Akaike’s ‘A’ Information Criterion</td>
<td>220.327619</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Schwarz’s Bayesian Criterion</td>
<td>221.728816</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RSquare</td>
<td>0.80027235</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RSquare Adj</td>
<td>0.80027235</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MAPE</td>
<td>15.8953925</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MAE</td>
<td>6.48774981</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2LogLikelihood</td>
<td>218.327619</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Parameter Estimates

| Term | Lag | Estimate | Std Error | t Ratio | Prob>|t| |
|------|-----|----------|-----------|---------|-------|
| MA1  | 1   | 0.22231599 | 0.1919162 | 1.16    | 0.2562 |

Forecast
Bad Debt Collections
Prewhitening Output

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-0.1606</td>
</tr>
<tr>
<td>-2</td>
<td>0.0903</td>
</tr>
<tr>
<td>-1</td>
<td>-0.1353</td>
</tr>
<tr>
<td>0</td>
<td>-0.1334</td>
</tr>
<tr>
<td>1</td>
<td>0.7236</td>
</tr>
<tr>
<td>2</td>
<td>-0.1029</td>
</tr>
<tr>
<td>3</td>
<td>-0.0284</td>
</tr>
<tr>
<td>4</td>
<td>-0.0774</td>
</tr>
</tbody>
</table>
Bad Debt Collections
Transfer Function Model Estimates

Model Summary

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>DF</td>
<td>27</td>
<td>Sum of Squared Errors</td>
<td>954.909308</td>
<td>Variance Estimate</td>
<td>35.3670083</td>
<td>Standard Deviation</td>
<td>5.94701676</td>
<td>Akaike’s 'A’ Information Criterion</td>
<td>187.823739</td>
</tr>
<tr>
<td>Schwarz’s Bayesian Criterion</td>
<td>190.558331</td>
<td>RSquare</td>
<td>0.62543276</td>
<td>RSquare Adj</td>
<td>0.61205536</td>
<td>MAPE</td>
<td>10.2670001</td>
<td>MAE</td>
<td>3.52031175</td>
</tr>
<tr>
<td>-2LogLikelihood</td>
<td>183.823739</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Parameter Estimates

| Variable          | Term     | Factor | Lag | Estimate | Std Error | t Ratio | Prob>|t| |
|-------------------|----------|--------|-----|----------|-----------|---------|--------|-------|
| OutstandingBadDebt | Scale    | 0      | 0   | 0.95066879 | 0.13594110 | 6.99    | <.0001*|
| BadDebtCollected  | MA1,1    | 1      | 1   | 0.41599759 | 0.18069910 | 2.30    | 0.0293*|

\[ (1 - B) \cdot \text{BadDebtCollected}_t = 0.9507 \cdot (1 - B) \cdot \text{OutstandingBadDebt}_{t-1} + (1 - 0.416 \cdot B) \cdot e_t \]
Bad Debt Collections
Transfer Function Model Residuals

Residual Value

Time

## Bad Debt Collections
### Transfer Function and Intervention Model Estimates

#### Model Summary

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>DF</td>
<td>25</td>
</tr>
<tr>
<td>Sum of Squared Errors</td>
<td>349.967178</td>
</tr>
<tr>
<td>Variance Estimate</td>
<td>13.9986765</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>3.74148079</td>
</tr>
<tr>
<td>Akaike's 'A' Information Criter.</td>
<td>163.431976</td>
</tr>
<tr>
<td>Schwarz's Bayesian Criterion</td>
<td>168.901159</td>
</tr>
<tr>
<td>RSquare</td>
<td>0.86272396</td>
</tr>
<tr>
<td>RSquare Adj</td>
<td>0.84688442</td>
</tr>
<tr>
<td>MAPE</td>
<td>4.7284338</td>
</tr>
<tr>
<td>MAE</td>
<td>2.21010938</td>
</tr>
<tr>
<td>-2LogLikelihood</td>
<td>155.431976</td>
</tr>
</tbody>
</table>

#### Parameter Estimates

| Variable                      | Term          | Factor | Lag | Estimate  | Std Error | t Ratio | Prob>|t| |
|-------------------------------|---------------|--------|-----|-----------|-----------|---------|-----|-----|
| OutstandingBadDebt:Scale      | 0             | 0      | 0   | 0.72381   | 0.074108  | 9.77    | <.0001*|
| Anticipation:Scale            | 0             | 0      | 0   | -20.38407 | 3.510718  | 5.77    | <.0001*|
| Change:Scale                  | 0             | 0      | 0   | -20.38407 | 2.965131  | -6.87   | <.0001*|
| BadDebtCollected:MA1:1        | 1             | 1      | 1   | 0.77240   | 0.130826  | 5.90    | <.0001*|

\[
(1 - B) \times \text{BadDebtCollected}_t = \left[ 0.72381 \times (1 - B) \times \text{OutstandingBadDebt}_{t-1} + 3.0549 \times (1 - B) \times \text{Anticipation}_t - 20.3841 \times (1 - B) \times \text{Change}_t + 0.7724 \times B \right] \times e_t
\]
Bad Debt Collections
Transfer Function and Intervention Model Residuals

Residual Value

Time

Models Fit to the Bad Debt Collections Data

- Model 1: IMA(1,1)

\[(1 - B)y_t = (1 - \theta_1 B)a_t\]

\[y_t = \frac{(1 - \theta_1 B)}{(1 - B)}a_t\]

- Model 2: IMA(1,1) with leading indicator

\[(1 - B)y_t = \omega_0 B(1 - B)x_t + (1 - \theta_1 B)a_t\]

\[y_t = \omega_0 x_{t-1} + \frac{(1 - \theta_1 B)}{(1 - B)}a_t\]

- Model 3: IMA(1,1) with leading indicator and intervention terms

\[(1 - B)y_t = \omega_0 B(1 - B)x_t + \omega_1 (1 - B)I_{1t} + \omega_2 (1 - B)I_{2t} + (1 - \theta_1 B)a_t\]

\[y_t = \omega_0 x_{t-1} + \omega_1 I_{1t} + \omega_2 I_{2t} + \frac{(1 - \theta_1 B)}{(1 - B)}a_t\]
Estimated Model Parameters for the Bad Debt Collections Data

- Model 1: IMA(1,1) univariate
  \[ y_t = \frac{(1 - 0.22B)}{(1 - B)}a_t \]

- Model 2: IMA(1,1) with leading indicator
  \[ y_t = 0.95x_{t-1} + \frac{(1 - 0.42B)}{(1 - B)}a_t \]

- Model 3: IMA(1,1) with leading indicator and intervention terms
  \[ y_t = 0.72x_{t-1} + 3.05I_{1t} - 20.38I_{2t} + \frac{(1 - 0.77B)}{(1 - B)}a_t \]
## Comparison of Models for Bad Debt Collections

<table>
<thead>
<tr>
<th>Model Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>IMA(1,1) Univariate</td>
<td>IMA(1,1) Transfer</td>
<td>IMA(1,1) Transfer Intervention</td>
</tr>
<tr>
<td><strong>θ₁</strong></td>
<td>0.22</td>
<td>0.42</td>
<td>0.77</td>
</tr>
<tr>
<td></td>
<td>(1.16)</td>
<td>(2.30)</td>
<td>(5.90)</td>
</tr>
<tr>
<td><strong>ω₀</strong></td>
<td>—</td>
<td>0.95</td>
<td>0.72</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(6.99)</td>
<td>(9.77)</td>
</tr>
<tr>
<td><strong>ω₁</strong></td>
<td>—</td>
<td>—</td>
<td>3.05</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.87)</td>
</tr>
<tr>
<td><strong>ω₂</strong></td>
<td>—</td>
<td>—</td>
<td>−20.38</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(−6.87)</td>
</tr>
<tr>
<td><strong>S</strong></td>
<td>9.36</td>
<td>5.95</td>
<td>3.74</td>
</tr>
<tr>
<td><strong>AIC&lt;sub&gt;c&lt;/sub&gt;</strong></td>
<td>220.3</td>
<td>187.8</td>
<td>163.4</td>
</tr>
<tr>
<td><strong>−2 log(Likelihood)</strong></td>
<td>218.3</td>
<td>183.8</td>
<td>155.4</td>
</tr>
<tr>
<td><strong>Ljung-Box χ²&lt;sub&gt;6&lt;/sub&gt;</strong></td>
<td>2.93</td>
<td>5.08</td>
<td>3.68</td>
</tr>
<tr>
<td><strong>Ljung-Box χ²&lt;sub&gt;6&lt;/sub&gt; p-value</strong></td>
<td>0.82</td>
<td>0.53</td>
<td>0.72</td>
</tr>
</tbody>
</table>
Module 12

Segment 4

Multivariate (Vector) Time Series Models
VARIMA Models
Vector Time Series Definition

• At each time period we observe a vector of \( m \) observations. At time \( t \) we observe \((Z_{1t}, Z_{2t}, \ldots, Z_{mt})\).

• Examples:
  
  - Daily minimum temperature in Kansas City, Des Moines, Chicago, Omaha, and Minneapolis.
  - Monthly total sales and advertising expenses for a company.

• There are many ways to express Vector Time Series models. The approach given here follows *Practical Experiences with Modeling and Forecasting Time Series* by G.M. Jenkins.
Vector Time Series Model

• We have $m$ time series that are interrelated, allowing for

$$Z_{1t} = f_1(Z_1, Z_2, \ldots, Z_m, a_1, a_2, \ldots, a_m) + a_{1t}$$
$$Z_{2t} = f_2(Z_1, Z_2, \ldots, Z_m, a_1, a_2, \ldots, a_m) + a_{2t}$$

$$\vdots$$

$$Z_{mt} = f_m(Z_1, Z_2, \ldots, Z_m, a_1, a_2, \ldots, a_m) + a_{mt}$$

where

• $Z_i$ is the complete realization of time series $i$.

• $a_i$ is the complete realization of “random shocks” for time series $i$. 
Vector ARIMA (VARIMA) Model

- The VARIMA model is

\[ \Phi(B)W_t = \theta_0 + \Theta(B)a_t \]

where

- \( a_t \) is the random shock \( m \) vector at time \( t \).
- \( Z_t \) is the time series \( m \) vector at time \( t \).
- \( W_t \) is the (possibly) transformed and (possibly) differenced working series \( m \) vector at time \( t \).
- \( \Phi(B) \) and \( \Theta(B) \) are \( m \times m \) matrices with polynomials in each element.
- \( \theta_0 \) is a vector of constant terms, which may be null for some time series (e.g., if there was differencing to get the working series \( W_t \)).
VARIMA Model Expanded

• The VARIMA model is

$$
\Phi(B)W_t = \theta_0 + \Theta(B)a_t
$$

where

$$
\phi_{ij}(B) = \begin{cases} 
1 - \phi_{1ij}B - \phi_{2ij}B^2 - \cdots - \phi_{pij}B^{pij}, & \text{if } i = j \\
-\phi_{1ij}B - \phi_{2ij}B^2 - \cdots - \phi_{pij}B^{pij}, & \text{otherwise}
\end{cases}
$$

• $\theta_{ij}(B)$ is defined in a manner that is similar to $\phi_{ij}(B)$.

• Need to choose $p_{i,j}$ and $q_{i,j}$, $i, j = 1, \ldots, m$ ($2 \times m^2$ choices).

• The model is stationary (invertible) if the roots of the determinantal equation $|\Phi(B)| (|\Theta(B)|)$ lie outside of the unit circle.
VARIMA Model Simple Special Case

- The VARIMA model for \( m = 2 \) with \( p_{ij} = q_{ij} = 1 \) for \( i, j = 1, 2 \) is

\[
\Phi(B)W_t = \theta_0 + \Theta(B) \alpha_t
\]

\[
\begin{bmatrix}
\phi_{11}(B) & \phi_{12}(B) \\
\phi_{21}(B) & \phi_{22}(B)
\end{bmatrix}
\begin{bmatrix}
w_{1t} \\
w_{2t}
\end{bmatrix}
= \begin{bmatrix}
\theta_{0,1} \\
\theta_{0,2}
\end{bmatrix} + \begin{bmatrix}
\theta_{11}(B) & \theta_{12}(B) \\
\theta_{21}(B) & \theta_{22}(B)
\end{bmatrix}
\begin{bmatrix}
\alpha_{1t} \\
\alpha_{2t}
\end{bmatrix}
\]

Replacing the polynomial operators with the polynomials gives:

\[
\begin{bmatrix}
1 - \phi_{111}B & -\phi_{112}B \\
-\phi_{121}B & 1 - \phi_{122}B
\end{bmatrix}
\begin{bmatrix}
w_{1t} \\
w_{2t}
\end{bmatrix}
= \begin{bmatrix}
\theta_{0,1} \\
\theta_{0,2}
\end{bmatrix} + \begin{bmatrix}
1 - \theta_{111}B & -\theta_{112}B \\
-\theta_{121}B & 1 - \theta_{122}B
\end{bmatrix}
\begin{bmatrix}
\alpha_{1t} \\
\alpha_{2t}
\end{bmatrix}
\]

After multiplying, applying the differencing operators, and rearranging we get the unscrambled model for the two time series:

\[
w_{1,t} = \phi_{111}w_{1,t-1} + \phi_{112}w_{2,t-1} \theta_{111}a_{1,t-1} - \theta_{112}a_{2,t-1} + \alpha_{1t}
\]

\[
w_{2,t} = \phi_{121}w_{1,t-1} + \phi_{122}w_{2,t-1} \theta_{121}a_{1,t-1} - \theta_{122}a_{2,t-1} + \alpha_{2t}
\]
Module 12

Segment 5

Relationship Among Variables Relating to Cattle Economics and VAR Models
Vector ARIMA Model for Cattle Economic Variables

  
  - $Z_{1t}$ Number of cattle slaughtered in Iowa in month $t$ (SL)
  - $Z_{2t}$ Number of cattle and calves placed on feed in Iowa in month $t$ (NF)
  - $Z_{3t}$ Index of meat prices received by farmers in month $t$ (MP)
  - $Z_{4t}$ Index of crop prices received by farmers in month $t$ (CP)

- Modeling, estimation, and predictions done with the SCA system (www.scausa.com) by Vicky Black (MS Project)
Fitted Vector ARIMA Model for Cattle Economic Variables

\[ SL_t = SL_{t-12} + 0.73(SL_{t-1} - SL_{t-13}) - 0.17(MP_{t-1} - MP_{t-13}) \\
- 0.61aSL_{t-12} + aSL_t \]

\[ NF_t = NF_{t-12} + 0.31(NF_{t-1} - NF_{t-13}) - 0.51(MP_{t-1} - MP_{t-13}) \\
+ 0.55(MP_{t-2} - MP_{t-14}) - 0.45(CP_{t-1} - CP_{t-13}) \\
- 1.4aMP_{t-12} + aNF_t \]

\[ MP_t = MP_{t-12} + 0.98(MP_{t-1} - MP_{t-13}) - 0.06aNF_{t-12} - 0.77aMP_{t-13} \\
- 0.11aCP_{t-12} + aMP_t \]

\[ CP_t = CP_{t-12} + 0.89(CP_{t-1} - CP_{t-13}) - 0.28(CP_{t-2} - CP_{t-14}) \\
- 0.28(CP_{t-3} - CP_{t-15}) + 0.09(SL_{t-3} - SL_{t-15}) \\
+ 0.15(MP_{t-3} - MP_{t-15}) - 0.99aCP_{t-12} + aCP_t \]

Note feedback relationship between

- Meat Price and New Feed
- Meat Price and Crop Price
Comparison of Multivariate and Univariate Model Predictions for NewFeed
Comparison of Multivariate and Univariate Model Predictions for NewFeed

1981 New Feed Predictions

- Multivariate
- Univariate
- Actual

Predictions

Jan | Mar | May | Jul | Sep | Nov

1981
Comparison of Multivariate and Univariate Model Predictions for Slaughter

1981 Slaughter Predictions

Actual
Multivariate
Univariate

Predictions

Jan Mar May Jul Sep Nov

1981
Comparison of Multivariate and Univariate Model Predictions for Slaughter

1981 Slaughter Predictions

Actual
Multivariate
Univariate

Predictions
Jan Mar May Jul Sep Nov

1981
Vector AR (VAR) Model

• Special case of VARIMA with only AR terms in the model

\[ Z_{1t} = f_{1}(Z_1, Z_2, \ldots, Z_m) + a_{1t} \]
\[ Z_{2t} = f_{2}(Z_1, Z_2, \ldots, Z_m) + a_{2t} \]
\[ \vdots \]
\[ Z_{mt} = f_{m}(Z_1, Z_2, \ldots, Z_m) + a_{mt} \]

• VAR models tend to not be parsimonious.
• VAR model parameters may be harder to interpret.
• With a sufficient amount of data, it is easier to identify a VAR model.
• More software is available for VAR modeling.
Software for Fitting VARIMA and VAR Models and other Time Series Models

• VARIMA Models
  ▶ R package fArma (in CRAN)
  ▶ SCA (www.scausa.com)

• VAR Models
  ▶ R package vars (in CRAN)
  ▶ SAS Econometrics and Time Series (ETS) (www.sas.com)
  ▶ STATA (www.stata.com)

• All of these packages also support univariate and transfer-function ARIMA models.