Module 5

Segment 1

$\text{AR}(p) \text{ Mean and AR}(1) \text{ Properties}$
Mean the AR(p) Model

Model: $Z_t = \theta_0 + \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t, \quad a_t \sim \text{nid}(0, \sigma_a^2)$

Mean: $\mu_Z \equiv \mathbb{E}(Z_t)$

\[
\mathbb{E}(Z_t) = \mathbb{E}(\theta_0 + \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t)
\]
\[
= \mathbb{E}(\theta_0) + \phi_1 \mathbb{E}(Z_{t-1}) + \cdots + \phi_p \mathbb{E}(Z_{t-p}) + \mathbb{E}(a_t)
\]
\[
= \theta_0 + (\phi_1 + \cdots + \phi_p) \mathbb{E}(Z_t)
\]
\[
= \frac{\theta_0}{1 - \phi_1 - \cdots - \phi_p}
\]

The last step requires several algebraic steps, but is straightforward.
Variance of the AR(1) Model

Model: \[ Z_t = \theta_0 + \phi_1 Z_{t-1} + a_t, \quad a_t \sim \text{nid}(0, \sigma_a^2) \]

Variance: \[ \gamma_0 \equiv \text{Var}(Z_t) \equiv E[(Z_t - \mu_Z)^2] = E(\dot{Z}^2) \]

\[
\begin{align*}
\gamma_0 &= E(\dot{Z}_t^2) \\
&= E[(\phi_1 \dot{Z}_{t-1} + a_t)^2] \\
&= E[(\phi_1^2 \dot{Z}_{t-1}^2 + 2\phi_1 \dot{Z}_{t-1}a_t + a_t^2)] \\
&= \phi_1^2 E(\dot{Z}_{t-1}^2) + 2\phi_1 E(\dot{Z}_{t-1}a_t) + E(a_t^2) \\
&= \phi_1^2 \gamma_0 + 0 + \sigma_a^2 \\
&= \frac{\sigma_a^2}{1-\phi_1^2}
\end{align*}
\]

or

\[
\text{Var}(Z_t) = \frac{\text{Var}(a_t)}{1 - \phi_1^2}
\]
Simulated AR(1) Data with $\phi_1 = 0.9, \sigma_a = 1$

Showing $\pm 2 \times \hat{\sigma}_Z$ limits

plot(arima.sim(n=250, model=list(ar=0.90)), ylab="")
Simulated AR(1) Data with $\phi_1 = -0.9, \sigma_a = 1$

Showing $\pm 2 \times \hat{\sigma}_Z$ limits

plot(arima.sim(n=250, model=list(ar=-0.90)), ylab="")
Autocovariance and Autocorrelation Functions for the AR(1) Model

**Autocovariance:**  
\[ \gamma_k \equiv \text{Cov}(Z_t, Z_{t+k}) \equiv E(\dot{Z}_t\dot{Z}_{t+k}) \]

\[ \gamma_1 = E(\dot{Z}_t\dot{Z}_{t+1}) = E[\dot{Z}_t(\phi_1 \dot{Z}_t + a_{t+1})] \]

\[ = \phi_1 E(\dot{Z}_t^2) + E(\dot{Z}_ta_{t+1}) \]

\[ = \phi_1 \gamma_0 + 0 = \phi_1 \gamma_0 \]

Thus \( \rho_1 = \frac{\gamma_1}{\gamma_0} = \phi_1 \).

\[ \gamma_2 = E(\dot{Z}_t\dot{Z}_{t+2}) = E[\dot{Z}_t(\phi_1 \dot{Z}_{t+1} + a_{t+2})] \]

\[ = \phi_1 E(\dot{Z}_t\dot{Z}_{t+1}) + E(\dot{Z}_ta_{t+2}) \]

\[ = \phi_1 \gamma_1 + 0 = \phi_1(\phi_1 \gamma_0) = \phi_1^2 \gamma_0 \]

Thus \( \rho_2 = \frac{\gamma_2}{\gamma_0} = \phi_1^2 \).

In general, for the AR(1) model, \( \rho_k = \frac{\gamma_k}{\gamma_0} = \phi_1^k \) for \(-1 < \phi_1 < 1\).
True ACF and PACF for AR(1) Model with $\phi_1 = 0.95$
True ACF and PACF for AR(1) Model with $\phi_1 = -0.95$
Simulated Realization \((\text{AR}(1), \phi_1 = 0.95, n = 75)\)

Graphical Output from Function \text{iden}
Simulated Realization \((AR(1), \phi_1 = 0.95, n = 300)\)

Graphical Output from Function \textit{idem}

Simulated data

Range-Mean Plot

ACF

PACF
Simulated Realization \((AR(1), \phi_1 = -0.95, n = 75)\)

Graphical Output from Function \texttt{iden}
Simulated Realization (AR(1), $\phi_1 = -0.95$, $n = 300$)

Graphical Output from Function `iden`

Simulated data

Range-Mean Plot

ACF

PACF
Module 5

Segment 2

AR(1) Stationarity Conditions
Using the Geometric Power Series to Re-express the AR(1) Model as an Infinite MA

\[(1 - \phi_1 B) \dot{Z}_t = a_t\]

\[\dot{Z}_t = (1 - \phi_1 B)^{-1} a_t\]

\[= (1 + \phi_1 B + \phi_1^2 B^2 + \cdots) a_t\]

\[= \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \cdots + a_t\]

Thus the AR(1) can be expressed as an infinite MA model.

If \(-1 < \phi_1 < 1\), then the weight on the old residuals is decreasing with age. This is the condition of “stationarity” for an AR(1) model.
Using the Back-substitution to Re-express the AR(1) Model as an Infinite MA

\[ \dot{Z}_t = \phi_1 \dot{Z}_{t-1} + a_t \]
\[ \dot{Z}_{t-1} = \phi_1 \dot{Z}_{t-2} + a_{t-1} \]
\[ \dot{Z}_{t-2} = \phi_1 \dot{Z}_{t-3} + a_{t-2} \]
\[ \dot{Z}_{t-3} = \phi_1 \dot{Z}_{t-4} + a_{t-3} \]

Substituting, successively, \( \dot{Z}_{t-1}, \dot{Z}_{t-2}, \dot{Z}_{t-3} \ldots \), shows that

\[ \dot{Z}_t = \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \phi_1^3 a_{t-3} + \cdots + a_t \]

This method works, more generally, for higher-order AR(\( p \)) models, but the algebra is tedious.
Notes on the AR(1) model

\[ Z_t = \phi_1 Z_{t-1} + a_t, \quad a_t \sim \text{nid}(0, \sigma_a^2) \]

- Because \( \rho_1 = \phi_1 \), we can estimate \( \phi_1 \) by \( \hat{\phi}_1 = \hat{\rho}_1 \).
- The root of \( (1 - \phi_1 B) = 0 \) is \( B = 1/\phi_1 \) and thus if \(-1 < \phi_1 < 1\), then the root is outside \([-1, 1]\) so that AR(1) will be stationary.
- \( \phi_1 = 1 \) implies \( Z_t = Z_{t-1} + a_t \), the “random walk” model.
- If \( \phi_1 > 1 \), then \( Z_t \) is explosive.
- If \( \phi_1 < -1 \), then \( Z_t \) is oscillating explosive.
True ACF and PACF for AR(1) Model with $\phi_1 = 0.99999$
Simulated Realization (AR(1), $\phi_1 = 0.99999, n = 75$)  
Graphical Output from Function iden
Simulated Realization \((\text{AR}(1), \phi_1 = 0.99999, n = 300)\)

Graphical Output from Function \textit{iden}
Re-expressing the AR($p$) Model as an Infinite MA

More generally, any AR($p$) model can be expressed as

\[ \phi_p(B) \dot{Z}_t = a_t \]

\[ \dot{Z}_t = \frac{1}{\phi_p(B)} a_t = \psi(B) a_t \]

\[ = \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots + a_t = \sum_{k=1}^{\infty} \psi_k a_{t-k} + a_t \]

Values of $\psi_1, \psi_2, \ldots$ depend on $\phi_1, \ldots, \phi_p$. For the AR model to be stationary, the $\psi_j$ values should not remain large as $j$ gets large. Formally, the stationarity condition is met if

\[ \sum_{j=1}^{\infty} |\psi_j| < \infty \]

Stationarity of an AR($p$) model can be checked by finding the roots of the polynomial $\phi_p(B) \equiv (1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p)$. All $p$ roots must lie outside of the “unit-circle.”
Module 5

Segment 3

AR(2) Stationarity and Model Properties
Checking the Stationarity of an AR(2) Model

Stationarity of an AR(2) model can be checked by finding the roots of the polynomial \((1 - \phi_1 B - \phi_2 B^2) = 0\). Both roots must lie outside of the "unit-circle."

\[
B = \frac{-\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}
\]

Roots have the form

\[
z = x + iy
\]

where \(i = \sqrt{-1}\). A root is "outside of the unit circle" if

\[
|z| = \sqrt{x^2 + y^2} > 1
\]

This method works for any \(p\), but for \(p > 2\) it is best to use numerical methods to find the \(p\) roots.
Checking the Stationarity of an AR(2) Model

Simple Rule

Both roots lying outside of the “unit-circle” implies

\[ \phi_2 + \phi_1 < 1 \quad \rightarrow \quad \phi_2 < 1 - \phi_1 \]

\[ \phi_2 - \phi_1 < 1 \quad \rightarrow \quad \phi_2 < 1 + \phi_1 \]

\[ -1 < \phi_2 < 1 \quad \rightarrow \quad -1 < \phi_2 < 1 \]

and defines the AR(2) triangle.
Autocovariance and Autocorrelation Functions for the AR(2) Model

\[ \gamma_1 \equiv \mathbb{E}(\dot{Z}_t \dot{Z}_{t+1}) = \mathbb{E}[\dot{Z}_t (\phi_1 \dot{Z}_t + \phi_2 \dot{Z}_{t-1} + a_{t+1})] \]
\[ = \phi_1 \mathbb{E}(\dot{Z}_t^2) + \phi_2 \mathbb{E}(\dot{Z}_t \dot{Z}_{t-1}) + \mathbb{E}(\dot{Z}_t a_{t+1}) \]
\[ = \phi_1 \gamma_0 + \phi_2 \gamma_1 + 0 \]

\[ \gamma_2 \equiv \mathbb{E}(\dot{Z}_t \dot{Z}_{t+2}) = \mathbb{E}[\dot{Z}_t (\phi_1 \dot{Z}_{t+1} + \phi_2 \dot{Z}_t + a_{t+2})] \]
\[ = \phi_1 \mathbb{E}(\dot{Z}_t \dot{Z}_{t+1}) + \phi_2 \mathbb{E}(\dot{Z}_t^2) + \mathbb{E}(\dot{Z}_t a_{t+2}) \]
\[ = \phi_1 \gamma_1 + \phi_2 \gamma_0 + 0 \]

Then using \( \rho_k = \frac{\gamma_k}{\gamma_0} \) and \( \rho_0 = 1 \), gives the AR(2) ACF

\[ \rho_1 = \phi_1 + \phi_2 \rho_1 \]
\[ \rho_2 = \phi_1 \rho_1 + \phi_2 \]
\[ \rho_3 = \phi_1 \rho_2 + \phi_2 \rho_1 \]
\[ \vdots \]
\[ \rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \]
True ACF and PACF for AR(2) Model with 
\( \phi_1 = 0.78, \phi_2 = 0.2 \)
True ACF and PACF for AR(2) Model with
$\phi_1 = 1, \phi_2 = -0.95$

True ACF

True PACF
Simulated Realization

(AR(2), $\phi_1 = 0.78, \phi_2 = 0.2, n = 75$)

Graphical Output from Function `iden`
Simulated Realization

(AR(2), $\phi_1 = 0.78$, $\phi_2 = 0.2$, $n = 300$)

Graphical Output from Function iden

Simulated data

Range-Mean Plot

ACF

PACF
Simulated Realization \((AR(2), \phi_1 = 1, \phi_2 = -0.95, n = 75)\)

Graphical Output from Function \texttt{iden}

Simulated data

Range-Mean Plot

ACF

PACF
Simulated Realization

\((AR(2), \phi_1 = 1, \phi_2 = -0.95, n = 300)\)

Graphical Output from Function \texttt{iden}
AR(2) Triangle Relating $\rho_1$ and $\rho_2$ to $\phi_1$ and $\phi_2$

Figure 6—Chart to Determine the Estimates of the Parameters of AR(2) Models
Figure 4—Correlation and Partial Correlation Functions of AR(2) Models
Module 5

Segment 4

Yule-Walker Equations and Their Applications and AR($p$) Variance
Yule-Walker Equations (Correlation Form)

AR(1) Yule-Walker Equation
\[ \rho_1 = \phi_1 \]

AR(2) Yule-Walker Equations
\[ \rho_1 = \phi_1 + \rho_1 \phi_2 \]
\[ \rho_2 = \rho_1 \phi_1 + \phi_2 \]

AR(3) Yule-Walker Equations
\[ \rho_1 = \phi_1 + \rho_1 \phi_2 + \rho_2 \phi_3 \]
\[ \rho_2 = \rho_1 \phi_1 + \phi_2 + \rho_1 \phi_3 \]
\[ \rho_3 = \rho_2 \phi_1 + \rho_1 \phi_2 + \phi_3 \]
Yule-Walker Equations (Correlation Form)

AR(4) Yule-Walker Equations

\[
\begin{align*}
\rho_1 &= \phi_1 + \rho_1 \phi_2 + \rho_2 \phi_3 + \rho_3 \phi_4 \\
\rho_2 &= \rho_1 \phi_1 + \phi_2 + \rho_1 \phi_3 + \rho_2 \phi_4 \\
\rho_3 &= \rho_2 \phi_1 + \rho_1 \phi_2 + \phi_3 + \rho_1 \phi_4 \\
\rho_4 &= \rho_3 \phi_1 + \rho_2 \phi_2 + \rho_1 \phi_3 + \phi_4
\end{align*}
\]

For given \(\rho_1, \ldots, \rho_4\), this is four linear equations with four unknowns.
\textbf{AR(}p\textbf{) Yule-Walker Equations (Correlation Form)}

\begin{align*}
\rho_1 &= \phi_1 + \rho_1\phi_2 + \rho_2\phi_3 + \cdots + \rho_{p-1}\phi_p \\
\rho_2 &= \rho_1\phi_1 + \phi_2 + \rho_1\phi_3 + \cdots + \rho_{p-2}\phi_p \\
&\vdots \\
\rho_p &= \rho_{p-1}\phi_1 + \rho_{p-2}\phi_2 + \rho_{p-3}\phi_3 + \cdots + \phi_p
\end{align*}

For given \(\rho_1, \ldots, \rho_p\), this is \(p\) linear equations with \(p\) unknowns.

Solving the Yule Walker equations for an AR(\(p\)) for \(\phi_p\) gives the PACF value \(\phi_{pp}\).
Applications of the Yule-Walker Equations

- Provides the true ACF of an AR(p) model (given $\phi_1, \phi_2, \ldots, \phi_p$, compute $\rho_1, \rho_2, \ldots \rho_p$ recursively).

- Provides estimate of $\phi_1, \phi_2, \ldots, \phi_p$ of an AR(p) model (given sample ACF values $\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_p$, substitute in Y-W equations and solve the set of simultaneous equations for $\hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_p$).

- Provides the sample PACF $\hat{\phi}_{1,1}, \hat{\phi}_{2,2}, \ldots, \hat{\phi}_{kk}$ for any realization. (Substitute $\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_k$ into the AR(k) Y-W equations and solve for $\hat{\phi}_k$). Same as Durbin’s formula.

- Provides the true PACF $\phi_{1,1}, \phi_{2,2}, \ldots, \phi_{kk}$ for any ARMA model. (Substitute $\rho_1, \rho_2, \ldots, \rho_k$ into the AR(k) Y-W equations and solve for $\phi_k$). Same as Durbin’s formula.
Variance of the AR\((p)\) model

\[
\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \phi_2 \dot{Z}_{t-2} + \cdots + \phi_p \dot{Z}_{t-p} + a_t
\]

Multiplying through by \(\dot{Z}_t\) gives

\[
\dot{Z}_t^2 = \phi_1 \dot{Z}_t \dot{Z}_{t-1} + \phi_2 \dot{Z}_t \dot{Z}_{t-2} + \cdots + \phi_p \dot{Z}_t \dot{Z}_{t-p} + \dot{Z}_t a_t
\]

Taking expectations and noting that \(E(\dot{Z}_t a_t) = \sigma_a^2\) gives

\[
\gamma_0 = \text{Var}(Z_t) = E(\dot{Z}_t^2) = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \cdots + \phi_p \gamma_p + \sigma_a^2
\]

Then dividing through by \(\gamma_0\) and solving for \(\gamma_0\) gives

\[
\gamma_0 = \frac{\sigma_a^2}{1 - \phi_1 \rho_1 - \phi_2 \rho_2 \cdots - \phi_p \rho_p}
\]

Note:

\[
E(\dot{Z}_t a_t) = E(\phi_1 \dot{Z}_{t-1} a_t + \phi_2 \dot{Z}_{t-2} a_t + \cdots + \phi_p \dot{Z}_{t-p} a_t + a_t^2) = \sigma_a^2
\]
Module 5

Segment 5

ARMA(1,1) Model Properties
Properties of the ARMA(1, 1) Model

\[
\phi_1(B)Z_t = \theta_0 + \theta_1(B)a_t \\
Z_t = \theta_0 + \phi_1 Z_{t-1} - \theta_1 a_{t-1} + a_t
\]

Noting that \( \mu_Z = E(Z_t) = \theta_0/(1 - \phi_1) \)

\[
\phi_1(B)\dot{Z}_t = \theta_1(B)a_t \\
\dot{Z}_t = \phi_1 \dot{Z}_{t-1} - \theta_1 a_{t-1} + a_t
\]

Omitting derivations,

\[
\gamma_0 = \text{Var}(Z_t) = \left[ (1 - 2\phi_1 \theta_1 + \theta_1^2)/(1 - \phi_1^2) \right] \sigma_a^2 \\
\rho_1 = (1 - \phi_1 \theta_1)(\phi_1 - \theta_1)/(1 + \theta_1^2 - 2\phi_1 \theta_1) \\
\rho_2 = \phi_1 \rho_1 \\
\rho_k = \phi_1 \rho_{k-1}
\]
True ACF and PACF for ARMA(1,1) Model with
\( \phi_1 = 0.95, \theta_1 = -0.95 \)
Simulated Realization

\((\text{ARMA}(1, 1), \phi_1 = 0.95, \theta_1 = -0.95, n = 75)\)

Graphical Output from Function `iden`
True ACF and PACF for ARMA(1, 1) Model with \( \phi_1 = -0.95, \theta_1 = 0.95 \)
Simulated Realization
(ARMA(1, 1), $\phi_1 = -0.95, \theta_1 = 0.95, n = 75$)

Graphical Output from Function `iden`

Simulated data

Range-Mean Plot

ACF

PACF
True ACF and PACF for ARMA(1, 1) Model with
\( \phi_1 = 0.9, \theta_1 = 0.5 \)
Simulated Realization

\((\text{ARMA}(1, 1), \phi_1 = 0.9, \theta_1 = 0.5, n = 75)\)

Graphical Output from Function \text{iden}

Simulated data

Range-Mean Plot

ACF

PACF
True ACF and PACF for ARMA(1, 1) Model with \( \phi_1 = -0.9, \theta_1 = -0.5 \)

**True ACF**

\[ \text{ma: -0.5 ar: -0.9} \]

**True PACF**

\[ \text{ma: -0.5 ar: -0.9} \]
Simulated Realization

(ARMA(1, 1), $\phi_1 = -0.9, \theta_1 = -0.5, n = 75$)

Graphical Output from Function `iden`
ARMA(1,1) Square Relating $\rho_1$ and $\rho_2$ to $\phi_1$ and $\theta_1$
ARMA(1,1) Square Showing ACF and PACF Functions

Figure 2—Correlation and Partial Correlation Functions of ARMA(1, 1) Models.
Module 5
Segment 6
Stationarity and Invertibility of the ARMA(1,1) Model
Stationarity of an ARMA(1,1) Model

\[(1 - \phi_1 B) \dot{Z}_t = (1 - \theta_1 B) a_t\]

\[\dot{Z}_t = \frac{(1 - \theta_1 B)}{(1 - \phi_1 B)} a_t\]

\[= (1 - \theta_1 B)(1 - \phi_1 B)^{-1} a_t\]

\[= (1 - \theta_1 B)(1 + \phi_1 B + \phi_1^2 B^2 + \cdots) a_t\]

\[= (\phi_1 - \theta_1) a_{t-1} + \phi_1 (\phi_1 - \theta_1) a_{t-2} + \phi_1^2 (\phi_1 - \theta_1) a_{t-3} + \cdots + a_t\]

\[= \sum_{j=1}^{\infty} \phi_1^{j-1}(\phi_1 - \theta_1) a_{t-j} + a_t = \sum_{j=0}^{\infty} \psi_j a_{t-j}\]

where \(\psi_j = \phi_1^{j-1}(\phi_1 - \theta_1)\) for \(j > 0\), \(\psi_0 = 1\), and \(\phi_1^0 \equiv 1\).

ARMA(1,1) model is stationary if \(-1 < \phi_1 < 1\).

ARMA(1,1) model is white noise (trivial model) if \(\phi_1 = \theta_1\).
Invertibility of an ARMA(1, 1) Model

\[(1 - \phi_1 B) \dot{Z}_t = (1 - \theta_1 B) a_t\]

\[a_t = \frac{(1 - \phi_1 B) \dot{Z}_t}{(1 - \theta_1 B)}\]

\[= (1 - \phi_1 B)(1 - \theta_1 B)^{-1} \dot{Z}_t\]

\[= (1 - \phi_1 B)(1 + \theta_1 B + \theta_1^2 B^2 + \cdots) \dot{Z}_t\]

\[= \dot{Z}_t + (\theta_1 - \phi_1) \dot{Z}_{t-1} + \theta_1(\theta_1 - \phi_1) \dot{Z}_{t-2} + \theta_1^2(\theta_1 - \phi_1) \dot{Z}_{t-3} + \cdots\]

\[\dot{Z}_t = (\phi_1 - \theta_1) \dot{Z}_{t-1} + \theta_1(\phi_1 - \theta_1) \dot{Z}_{t-2} + \theta_1^2(\phi_1 - \theta_1) \dot{Z}_{t-3} + \cdots + a_t\]

\[= \sum_{j=1}^{\infty} \theta_1^{j-1}(\phi_1 - \theta_1) \dot{Z}_{t-j} + a_t = \sum_{j=1}^{\infty} \pi_j \dot{Z}_{t-j} + a_t\]

where \(\pi_j = \theta_1^{j-1}(\phi_1 - \theta_1), j > 0\) and \(\pi_0 = 1\).

ARMA(1,1) model is invertible if \(-1 < \theta_1 < 1\).
True ACF and PACF for ARMA(1, 1) Model with \( \phi_1 = 0.9, \theta_1 = 0.9 \)

**True ACF**

\[
\text{ma: 0.9 ar: 0.9}
\]

**True PACF**

\[
\text{ma: 0.9 ar: 0.9}
\]
Simulated Realization

(ARMA(1, 1), $\phi_1 = 0.9, \theta_1 = 0.9, n = 75$)

Graphical Output from Function `iden`
Properties of ARMA\((p, q)\) model

\[
\phi_p(B)Z_t = \theta_0 + \theta_q(B)a_t
\]

\[
\phi_p(B)\dot{Z}_t = \theta_q(B)a_t
\]

\[
(1 - \phi_1B - \phi_2B^2 - \cdots - \phi_pB^p)\dot{Z}_t = (1 - \theta_1B - \theta_2B^2 - \cdots - \theta_qB^q)a_t
\]

- \(\mu_Z \equiv E(Z_t) = \frac{\theta_0}{1 - \phi_1 - \cdots - \phi_p}\)

- An ARMA\((p, q)\) model is **stationary** if all of the roots of \(\phi_p(B)\) lie **outside** of the unit circle.

- An ARMA\((p, q)\) model is **invertible** if all of the roots of \(\theta_q(B)\) lie **outside** of the unit circle.
Expressing an ARMA\((p, q)\) Model as an Infinite MA

\[
\phi_p(B) \dot{Z}_t = \theta_q(B) a_t
\]

\[
\dot{Z}_t = [\phi_p(B)]^{-1} \theta_q(B) a_t = \psi(B) a_t
\]

\[
\phi_p(B) \psi(B) = \theta_q(B)
\]

- Given \(\phi_p(B)\) and \(\theta_q(B)\), solve for \(\psi(B)\), giving \(\psi_1, \psi_2, \ldots\).

- Similar method for expressing an ARMA\((p, q)\) model as an infinite AR

\[
\phi_p(B) = \pi(B) \theta_q(B)
\]

Given \(\phi_p(B)\) and \(\theta_q(B)\), solve for \(\pi(B)\), giving \(\pi_1, \pi_2, \ldots\).
Expressing and ARMA\((p, q)\) Model as an Infinite MA

Using ARMA(2,2) as a particular example

\[
\phi_2(B)\psi(B) = \theta_2(B)
\]

\[
(1 - \phi_1B - \phi_2B^2)(1 + \psi_1B + \psi_2B^2 + \psi_3B^3 + \cdots) = (1 - \theta_1B - \theta_2B^2)
\]

Multiplying out gives

\[
1 + \psi_1B + \psi_2B^2 + \psi_3B^3 + \cdots
\]

\[
-\phi_1B - \psi_1\phi_1B^2 - \psi_2\phi_1B^3 - \cdots
\]

\[
-\phi_2B^2 - \psi_1\phi_2B^3 - \cdots = 1 - \theta_1B - \theta_2B^2
\]

Equating terms with the same power of B gives

\[
B : \quad \psi_1 - \phi_1 = -\theta_1 \quad \rightarrow \quad \psi_1 = -\theta_1 + \phi_1
\]

\[
B^2 : \quad \psi_2 - \psi_1\phi_1 - \phi_2 = -\theta_2 \quad \rightarrow \quad \psi_2 = -\theta_2 + \psi_1\phi_1 + \phi_2
\]

\[
B^3 : \quad \psi_3 - \psi_2\phi_1 - \psi_1\phi_2 = 0 \quad \rightarrow \quad \psi_3 = \psi_2\phi_1 + \psi_1\phi_2
\]

\[
\vdots
\]

\[
B^k : \quad \psi_k - \psi_{k-1}\phi_1 - \psi_{k-2}\phi_2 = 0 \quad \rightarrow \quad \psi_k = \psi_{k-1}\phi_1 + \psi_{k-2}\phi_2
\]
Module 5

Segment 7

ARMA Identification Exercises
Simulated Time Series #5
Graphical Output from Function \texttt{iden(simsta5.d)}
Simulated Time Series #6
Graphical Output from Function `iden`

Range-Mean Plot

Simulated Time Series #6
\( w = \text{Sales} \)

ACF

PACF
Simulated Time Series #7
Graphical Output from Function `iden`

Range-Mean Plot

ACF

PACF
Simulated Time Series #8
Graphical Output from Function \texttt{iden}

Simulated Time Series #8

w = decibles

Range-Mean Plot

ACF

PACF

\texttt{5 - 64}
Simulated Time Series #11
Graphical Output from Function `iden`

![Simulated Time Series #11 Graphical Output](image)

**Range-Mean Plot**
- Mean
- Range
- Time

**ACF**
- Lag
- ACF
- Lag

**PACF**
- Lag
- Partial ACF
- Lag
Simulated Time Series #13
Graphical Output from Function \texttt{iden}

Simulated Time Series #13

\texttt{w= Pressure}

Range-Mean Plot

ACF

PACF
Simulated Time Series #15
Graphical Output from Function iden

Simulated Time Series #15

\( w \) Pressure

ACF

\( \text{Lag} \)

\[ \text{ACF} \]

0 10 20 30

-1.0 0.0 0.5 1.0

PACF

\( \text{Lag} \)

\[ \text{Partial ACF} \]

0 10 20 30

-1.0 0.0 0.5 1.0