Module 4

Stationary Time Series Models Part 1
MA Models and Their Properties

Class notes for Statistics 451: Applied Time Series
Iowa State University
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20h 58min
Module 4

Segment 1

ARMA Notation, Conventions, Expectations, and other Preliminaries
Deviations from the Sample Mean

- The sample mean is computed as
  \[ \hat{\mu} = \bar{z} = \frac{\sum_{t=1}^{n} z_t}{n} \]

- We use the centered realization \( z_t - \bar{z} \) in the computation of many statistics. For example
  \[ \hat{\sigma}_Z = \hat{\gamma}_0 = S_Z = \sqrt{\frac{\sum_{t=1}^{n} (z_t - \bar{z})^2}{n - 1}} \]

- Also used in computing sample autocovariances and autocorrelations. Subtracting out the mean (\( \bar{z} \)) does not change the sample variance (\( \hat{\gamma}_0 \)), sample autocovariances (\( \hat{\gamma}_k \)), or sample autocorrelations (\( \hat{\rho}_k \)) of a realization.
Change in Business Inventories 1955-1969 and Centered Change in Business Inventories 1955-1969

Change in Inventories

Centered Change in Inventories
Deviations from the Process Mean

- The stochastic process is $Z_1, Z_2, \ldots$.
- The process mean is $\mu = E(Z_t)$.
- Some derivations are simplified by using $\dot{Z}_t = Z_t - \mu$ to compute model properties (only the mean has changed).
- In particular,

$$E(\dot{Z}_t) = E(Z_t - \mu) = E(Z_t) - E(\mu) = \mu - \mu = 0$$

Similarly,

$$\gamma_0 = \text{Var}(\dot{Z}_t) = \text{Var}(Z_t - \mu) = \text{Var}(Z_t) + \text{Var}(\mu) = \text{Var}(Z_t) = \sigma_Z^2$$

and

$$\gamma_k = \text{Cov}(\dot{Z}_t, \dot{Z}_{t+k}) = \text{Cov}(Z_t - \mu, Z_{t+k} - \mu) = \text{Cov}(Z_t, Z_{t+k})$$

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$
Backshift Operator Notation

Backshift operators:
\[ B Z_t = Z_{t-1} \]
\[ B^2 Z_t = BBZ_t = BZ_{t-1} = Z_{t-2} \]
\[ B^3 Z_t = Z_{t-3}, \text{ etc.} \]

If \( C \) is a constant, then \( BC = C \)

Differencing operators:
\[ (1 - B)Z_t = Z_t - BZ_t = Z_t - Z_{t-1} \]
\[ (1 - B)^2 Z_t = (1 - 2B + B^2)Z_t = Z_t - 2Z_{t-1} + Z_{t-2} \]

Seasonal differencing operators:
\[ (1 - B^4)Z_t = Z_t - B^4Z_t = Z_t - Z_{t-4} \]
\[ (1 - B^{12})Z_t = Z_t - B^{12}Z_t = Z_t - Z_{t-12} \]
Polynomial Operator Notation

\[ \phi(B) = \phi_p(B) \equiv (1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) \]
is useful for expressing the AR\((p)\) model.

For example, with \( p = 2 \), an AR(2) model.

\[ \phi_2(B) \dot{Z} = \dot{a} \]
\[ (1 - \phi_1 B - \phi_2 B^2) \dot{Z} = \dot{a} \]
\[ \dot{Z} - \phi_1 \dot{Z}_{t-1} - \phi_2 \dot{Z}_{t-2} = \dot{a} \]
or
\[ \dot{Z} = \phi_1 \dot{Z}_{t-1} + \phi_2 \dot{Z}_{t-2} + \dot{a} \]

Other polynomial operators:
\[ \theta(B) = \theta_q(B) \equiv (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q) \] [as in \( \dot{Z} = \theta_q(B) \dot{a} \)]
\[ \psi(B) = \psi_\infty(B) \equiv (1 + \psi_1 B + \psi_2 B^2 + \cdots) \] [as in \( \dot{Z} = \psi(B) \dot{a} \)]
\[ \pi(B) = \pi_\infty(B) \equiv (1 - \pi_1 B - \pi_2 B^2 - \cdots) \] [as in \( \pi(B) \dot{Z} = \dot{a} \)]
General ARMA\((p, q)\) Model in Terms of \(\dot{Z}_t\)

\[
\begin{align*}
\dot{Z}_t & \equiv Z_t - \mu \\
\phi_p(B)\dot{Z}_t & = \theta_q(B)a_t \\
(1 - \phi_1B - \phi_2B^2 - \cdots - \phi_pB^p)\dot{Z}_t & = \\
(1 - \theta_1B - \theta_2B^2 - \cdots - \theta_qB^q)a_t & \\
\dot{Z}_t - \phi_1\dot{Z}_{t-1} - \phi_2\dot{Z}_{t-2} - \cdots - \phi_p\dot{Z}_{t-p} & = \\
a_t - \theta_1a_{t-1} - \theta_2a_{t-2} - \cdots - \theta_qa_{t-q} & \\
or & \\
\dot{Z}_t & = \phi_1\dot{Z}_{t-1} + \phi_2\dot{Z}_{t-2} + \cdots + \phi_p\dot{Z}_{t-p} - \theta_1a_{t-1} - \theta_2a_{t-2} - \cdots - \theta_qa_{t-q} + a_t
\end{align*}
\]
Mean ($\mu$) and Constant Term ($\theta_0$) for an ARMA(2,2) Model

Replace $\dot{Z}_t$ with $\dot{Z}_t = Z_t - \mu$ and solve for $Z_t = \cdots$.

For example, using $p = 2$ and $q = 2$, giving an ARMA(2,2) model:

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \phi_2 \dot{Z}_{t-2} - \theta_1 a_{t-1} - \theta_2 a_{t-2} + a_t$$

and

$$Z_t - \mu = \phi_1 (Z_{t-1} - \mu) + \phi_2 (Z_{t-2} - \mu) - \theta_1 a_{t-1} - \theta_2 a_{t-2} + a_t$$
$$Z_t = \mu - \phi_1 \mu - \phi_2 \mu + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} - \theta_1 a_{t-1} - \theta_2 a_{t-2} + a_t$$
$$Z_t = \theta_0 + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} - \theta_1 a_{t-1} - \theta_2 a_{t-2} + a_t$$

where $\theta_0 = \mu - \phi_1 \mu - \phi_2 \mu = \mu (1 - \phi_1 - \phi_2)$ is the ARMA model "constant term." Also, $E(Z_t) = \mu = \theta_0 / (1 - \phi_1 - \phi_2)$.
Note that $E(Z_t) = \mu = \theta_0$ for an MA($q$) model.
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Segment 2

Properties of the MA(1) Model and the PACF
Simulated MA(1) Data with $\theta_1 = 0.9$, $\sigma_a = 1$

plot(arima.sim(n=250, model=list(ma=0.90)), ylab="")
Simulated MA(1) Data with $\theta_1 = -0.9$, $\sigma_a = 1$

plot(arima.sim(n=250, model=list(ma=-0.90)), ylab="")
Mean and Variance of the MA(1) Model

Model: \( Z_t = \theta_0 - \theta_1 a_{t-1} + a_t, \quad a_t \sim \text{nid}(0, \sigma_a^2) \)

Mean: \( \mu_Z \equiv E(Z_t) = E(\theta_0 - \theta_1 a_{t-1} + a_t) = \theta_0 + 0 + 0 = \theta_0 \).

Variance: \( \gamma_0 \equiv \sigma_{Z_t}^2 \equiv \text{Var}(Z_t) \equiv E[(Z_t - \mu_Z)^2] = E(\dot{Z}_t^2) \)
\[
\gamma_0 = E(\dot{Z}_t^2) \\
= E[(-\theta_1 a_{t-1} + a_t)^2] \\
= E[(\theta_1^2 a_{t-1} + 2\theta_1 a_{t-1} a_t + a_t^2)] \\
= \theta_1^2 E(a_{t-1}^2) - 2\theta_1 E(a_{t-1} a_t) + E(a_t^2) \\
= \theta_1^2 \sigma_a^2 - 0 + \sigma_a^2 \\
= (\theta_1^2 + 1)\sigma_a^2
\]
Autocovariance and Autocorrelation Functions for the MA(1) Model

Autocovariance:

\[ \gamma_k \equiv \text{Cov}(Z_t, Z_{t+k}) = E[(Z_t - \mu_Z)(Z_{t+k} - \mu_Z)] = E(\dot{Z}_t \dot{Z}_{t+k}) \]

\[ \gamma_1 \equiv E(\dot{Z}_t \dot{Z}_{t+1}) = E[(-\theta_1 a_{t-1} + a_t)(-\theta_1 a_t + a_{t+1})] \]

\[ = E(\theta_1^2 a_{t-1} a_t - \theta_1 a_{t-1} a_{t+1} - \theta_1 a_t^2 + a_t a_{t+1}) \]

\[ = 0 - 0 - \theta_1 \sigma_a^2 + 0 \]

\[ = -\theta_1 \sigma_a^2 \]

Thus \( \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{-\theta_1}{1+\theta_1^2}. \)

Using similar operations, it is easy to show that \( \gamma_2 \equiv E(\dot{Z}_t \dot{Z}_{t+2}) = 0 \) and thus \( \rho_2 = \gamma_2/\gamma_0 = 0. \) In general, for MA(1), \( \rho_k = \frac{\gamma_k}{\gamma_0} = 0 \) for \( k > 1. \)
Partial Autocorrelation Function

For any model, the process PACF $\phi_{kk}$ can be computed from the process ACF from $\rho_1, \rho_2, \ldots$ using the recursive formula from Durbin (1960):

$$
\begin{align*}
\phi_{1,1} &= \rho_1 \\
\phi_{kk} &= \rho_k - \sum_{j=1}^{k-1} \phi_{k-1,j} \rho_{k-j}, \quad k = 2, 3, \ldots \\
&= \frac{\rho_k - \sum_{j=1}^{k-1} \phi_{k-1,j} \rho_{k-j}}{1 - \sum_{j=1}^{k-1} \phi_{k-1,j} \rho_j}, \quad k = 2, 3, \ldots
\end{align*}
$$

where

$$
\phi_{kj} = \phi_{k-1,j} - \phi_{kk} \phi_{k-1,k-j} \quad (k = 3, 4, \ldots; \ j = 1, 2, \ldots, k - 1)
$$

This formula is based on the solution of the Yule-Walker equations (covered later).

Also, the sample PACF can be computed from the sample ACF. That is, $\hat{\phi}_{kk}$ is a function of $\hat{\rho}_1, \hat{\rho}_2, \ldots$. 

Hints For Using Durbin’s formula

\[ \phi_{11} = \rho_1 \]

depends on \( \rho_1 \)

\[ \phi_{22} = \frac{\rho_2 - \phi_{11}\rho_1}{1 - \phi_{11}\rho_1} \]

depends on \( \rho_1, \rho_2 \) and \( \phi_{11} \)

- \( \phi_{33} \) depends on \( \rho_1, \rho_2, \rho_3, \phi_{21}, \) and \( \phi_{22} \)

- \( \phi_{44} \) depends on \( \rho_1, \rho_2, \rho_3, \rho_4, \phi_{31}, \phi_{32}, \) and \( \phi_{33} \)

- \( \phi_{55} \) depends on \( \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \phi_{41}, \phi_{42}, \phi_{43}, \) and \( \phi_{44} \)

- and so on.
True ACF and PACF for MA(1) Model with $\theta_1 = 0.95$
True ACF and PACF for MA(1) Model with $\theta_1 = -0.95$
Simulated Realization \((\text{MA}(1), \theta_1 = 0.95, n = 75)\)

Graphical Output from Function \texttt{iden}

Simulated data

Range-Mean Plot

ACF

PACF
Simulated Realization (MA(1), \( \theta_1 = 0.95, n = 300 \))

Graphical Output from Function `iden`

Simulated data

Range-Mean Plot

ACF

PACF
Simulated Realization \( (MA(1), \theta_1 = -0.95, n = 75) \)

Graphical Output from Function \texttt{iden}
Simulated Realization \((\text{MA}(1), \theta_1 = -0.95, n = 300)\)

Graphical Output from Function `iden`
Module 4

Segment 3

Inverting the MA(1) Model and MA(q) Properties
Geometric Power Series

\[
\frac{1}{1 - \theta_1 B} = (1 - \theta_1 B)^{-1} = (1 + \theta_1 B + \theta_1^2 B^2 + \cdots)
\]

This expansion provides a convenient method for re-expressing parts of some ARMA models. When \(|\theta_1| < 1\) the series converges.

Similarly,

\[
\frac{1}{1 - \phi_1 B} = (1 - \phi_1 B)^{-1} = (1 + \phi_1 B + \phi_1^2 B^2 + \cdots)
\]

Again, when \(|\phi_1| < 1\) the series converges.
Using the Geometric Power Series to Re-express the MA(1) Model as an Infinite AR

\[ \dot{Z}_t = (1 - \theta_1 B) a_t \]
\[ a_t = (1 - \theta_1 B)^{-1} \dot{Z}_t \]
\[ a_t = (1 + \theta_1 B + \theta_1^2 B^2 + \cdots) \dot{Z}_t \]
\[ \dot{Z}_t = -\theta_1 \dot{Z}_{t-1} - \theta_1^2 \dot{Z}_{t-2} - \cdots + a_t \]

Thus the MA(1) can be expressed as an infinite AR model.

If \(-1 < \theta_1 < 1\), then the weight on the old observations is decreasing with age (having more practical meaning). This is the condition of “invertibility” for an MA(1) model.
Using the Back-substitution to Re-express the MA(1) Model as an Infinite AR

\[ \dot{Z}_t = -\theta_1 a_{t-1} + a_t \]

\[ a_{t-1} = \dot{Z}_{t-1} + \theta_1 a_{t-2} \]

\[ a_{t-2} = \dot{Z}_{t-2} + \theta_1 a_{t-3} \]

\[ a_{t-3} = \dot{Z}_{t-3} + \theta_1 a_{t-4} \]

Substituting, successively, \( a_{t-1}, a_{t-2}, a_{t-3}, \ldots \), shows that

\[ \dot{Z}_t = -\theta_1 \dot{Z}_{t-1} - \theta_1^2 \dot{Z}_{t-2} - \cdots + a_t \]

This method works, more generally, for higher-order MA(\( q \)) models, but the algebra is tedious.
Notes on the MA(1) Model

Given $\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{-\theta_1}{(1+\theta_1^2)}$ we can see

- $-0.5 \leq \rho_1 \leq 0.5$
- Solving the $\rho_1 = \cdots$ quadratic equation for $\theta_1$ gives

$$\theta_1 = \begin{cases} \frac{-1}{2\rho_1} \pm \sqrt{\frac{1}{(2\rho_1)^2} - 1} & \rho_1 \neq 0 \\ 0 & \rho_1 = 0 \end{cases}$$

The two solutions are related by

$$\theta_1 = \frac{1}{\theta_1'}$$

For any $-0.5 \leq \rho_1 \leq 0.5$, the solution with $-1 < \theta_1 < 1$ is the "invertible" parameter value.

- Substituting $\hat{\rho}_1$ for $\rho_1$ provides an estimator for $\theta_1$. 
Mean, Variance, and Covariance of the MA(q) Model

Model: \( Z_t = \theta_0 + \theta_q(B) a_t = \theta_0 - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t \)

Mean: \( \mu_Z \equiv E(Z_t) = E(\theta_0 - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t) = \theta_0 \).

Variance: \( \gamma_0 \equiv \text{Var}(Z_t) \equiv E[(Z_t - \mu_Z)^2] = E(\dot{Z}^2) \)

Centered MA(q): \( \dot{Z}_t = \theta_q(B) a_t = -\theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t \)

\[
\text{Var}(Z_t) \equiv \gamma_0 = E(\dot{Z}_t^2) = (1 + \theta_1^2 + \cdots + \theta_q^2) \sigma_a^2
\]

\[
\text{Cov}(Z_t, Z_{t+k}) = \gamma_k = E(\dot{Z}_t \dot{Z}_{t+k}) = \cdots
\]

\[
\rho_k = \frac{\gamma_k}{\gamma_0}
\]
Re-expressing the MA(\(q\)) Model as an Infinite AR

More generally, any MA(\(q\)) model can be expressed as

\[
\dot{Z}_t = \theta_q(B) a_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B) a_t
\]

\[
\frac{1}{\theta_q(B)} \dot{Z}_t = \pi(B) \dot{Z}_t = (1 - \pi_1 B - \pi_2 B^2 - \cdots) \dot{Z}_t = a_t
\]

\[
\dot{Z}_t = \pi_1 \dot{Z}_{t-1} + \pi_2 \dot{Z}_{t-2} + \cdots + a_t = \sum_{k=1}^{\infty} \pi_k \dot{Z}_{t-k} + a_t
\]

Values of \(\pi_1, \pi_2, \ldots\) depend on \(\theta_1, \ldots, \theta_q\). For the model to have practical meaning, the \(\pi_j\) values should not remain large as \(j\) gets large. Formally, the invertibility condition is met if

\[
\sum_{i=1}^{\infty} |\pi_j| < \infty
\]

Invertibility of an MA(\(q\)) model can be checked by finding the roots of the polynomial \(\theta_q(B) \equiv (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q) = 0\). All \(q\) roots must lie outside of the “unit-circle.”
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Segment 4

Checking Invertibility and Polynomial Roots
Checking the Invertibility of an MA(2) Model

Invertibility of an MA(2) model can be checked by finding the roots of the polynomial \( \theta_2(B) = (1 - \theta_1 B - \theta_2 B^2) = 0 \). Both roots must lie outside of the “unit-circle.”

\[
B = \frac{-\theta_1 \pm \sqrt{\theta_1^2 + 4\theta_2}}{2\theta_2}
\]

Roots have the form

\[
z = x + iy
\]

where \( i = \sqrt{-1} \). A root is “outside of the unit circle” if

\[
|z| = \sqrt{x^2 + y^2} > 1
\]

This method works for any \( q \), but for \( q > 2 \) it is best to use numerical methods to find the \( q \) roots.
Roots of \( (1 - 1.5B - 0.4B^2) = 0 \)

\texttt{arma.root(c(1.5, 0.4))}

Coefficients= 1.5, 0.4

Polynomial Function versus B

Roots and the Unit Circle
Roots of the Quadratic \((1 - 0.5B + 0.9B^2) = 0\) and the Unit Circle

Unit Circle
Coefficients = 0.5, -0.9
Roots of \((1 - 0.5B + 0.9B^2) = 0\)

```
arma.root(c(0.5, -0.9))
```

Coefficients = 0.5, -0.9

Polynomial Function versus B

![Polynomial Function Graph](image)

Roots and the Unit Circle

![Roots on the Unit Circle](image)
Roots of \((1 - 0.5B + 0.9B^2 - 0.1B^3 - 0.5B^4) = 0\)

\[\text{arma.root}(c(0.5, -0.9, 0.1, 0.5))\]

Coefficients = 0.5, -0.9, 0.1, 0.5

Polynomial Function versus B

Roots and the Unit Circle
Checking the Invertibility of an MA(2) Model
Simple Rule

Both roots lying outside of the “unit-circle” implies

\[ \theta_2 + \theta_1 < 1 \quad \theta_2 < 1 - \theta_1 \]
\[ \theta_2 - \theta_1 < 1 \quad \theta_2 < 1 + \theta_1 \]
\[ -1 < \theta_2 < 1 \quad -1 < \theta_2 < 1 \]

and the inequalities define the MA(2) triangle.
Module 4

Segment 5

MA(2) Model Properties
Autocovariance and Autocorrelation Functions
for the MA(2) Model

\[ \gamma_1 \equiv \text{Cov}(Z_t, Z_{t+1}) = \mathbb{E}[(Z_t - \mu_Z)(Z_{t+1} - \mu_Z)] = \mathbb{E}(\dot{Z}_t \dot{Z}_{t+1}) \]

\[ = \mathbb{E}((-\theta_1 a_{t-1} - \theta_2 a_{t-2} + a_t)(-\theta_1 a_t - \theta_2 a_{t-1} + a_{t+1})) \]

\[ = \mathbb{E}(\theta_1 \theta_2 a_{t-1}^2 - \theta_1 a_t^2) + 0 + 0 + 0 + 0 + 0 + 0 + 0 \]

\[ = \theta_1 \theta_2 \mathbb{E}(a_{t-1}^2) - \theta_1 \mathbb{E}(a_t^2) = (\theta_1 \theta_2 - \theta_1)\sigma_a^2 \]

Thus

\[ \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta_1 \theta_2 - \theta_1}{1 + \theta_1^2 + \theta_2^2}. \]

Using similar operations, it is easy to show that

\[ \rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} \]

and that, in general, for MA(2), \( \rho_k = \frac{\gamma_k}{\gamma_0} = 0 \) for \( k > 2 \).
True ACF and PACF for MA(2) Model with

\[ \theta_1 = 0.78, \theta_2 = 0.2 \]
True ACF and PACF for MA(2) Model with
\( \theta_1 = 1, \theta_2 = -0.95 \)
Simulated Realization (MA(2), $\theta_1 = 0.78, \theta_2 = 0.2, n = 75$)

Graphical Output from Function `iden`

Simulated data

<table>
<thead>
<tr>
<th>Mean</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.04</td>
<td>3.8</td>
</tr>
<tr>
<td>-0.02</td>
<td>4.0</td>
</tr>
<tr>
<td>0.02</td>
<td>4.2</td>
</tr>
<tr>
<td>0.04</td>
<td>4.4</td>
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<tr>
<td>0.06</td>
<td>4.6</td>
</tr>
<tr>
<td>0.08</td>
<td>4.8</td>
</tr>
</tbody>
</table>

ACF

-1.0 0.0 0.5 1.0

Lag

PACF

-1.0 0.0 0.5 1.0

Lag
Simulated Realization

\((\text{MA}(2), \theta_1 = 0.78, \theta_2 = 0.2, n = 300)\)

Graphical Output from Function \textit{idem}
Simulated Realization \((MA(2), \theta_1 = 1, \theta_2 = -0.95, n = 75)\)

Graphical Output from Function \texttt{iden}
Simulated Realization

\((MA(2), \theta_1 = 1, \theta_2 = -0.95, n = 300)\)

Graphical Output from Function `iden`
MA(2) Triangle Relating $\rho_1$ and $\rho_2$ to $\theta_1$ and $\theta_2$

Figure 3—Chart to Find the Initial Estimates of the Parameters of MA(1) and MA(2) Models
MA(2) Triangle Showing ACF and PACF Functions

Figure 1—Correlation and Partial Correlation Functions of MA(2) Models.