Statistical Methods for Probabilistic Design

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Overview

- Probabilistic design background and concerns.
- Degradation failure and reliability modeling.
- Variability and uncertainty—relationship to reliability.
- Motivating example–jet engine fan disk reliability.
- Simplified jet engine fan disk reliability model.
- Using data to reduce uncertainty.
- Economic choice of experiments to provide needed information for product design.

Probabilistic Design Background

- Today's manufacturers face strong pressure to improve performance/cost ratios of their products.
- Traditional deterministic engineering design tends use large (conservative) safety factors.
- Managers are concerned with the cost of being "too conservative."
- Probabilistic design is being suggested/considered/used as an alternative.
- Probabilistic design promises lower weight, lower cost, and higher performance for products like jet engine components, automobiles, transmissions, telecommunications systems, buildings and bridges, washing machines, etc.
- Possible advantage: allows putting safety factors where they are most needed (protect against things unknown).

Pratt and Whitney Probabilistic Design System (PDS)

"The goal of the PDS program is to produce lower weight [jet engine] components by replacing well proven, but conservative, deterministic methods with probabilistic design techniques."

J.D. Adamson "The Probabilistic Design System Development Experience," *Proceedings of the 35th AIAA/ASME/ASCE/AHS/A Structures, Structural Dynamics, and Materials Conference*, AIAA-94-1444, April 1994.

Concerns

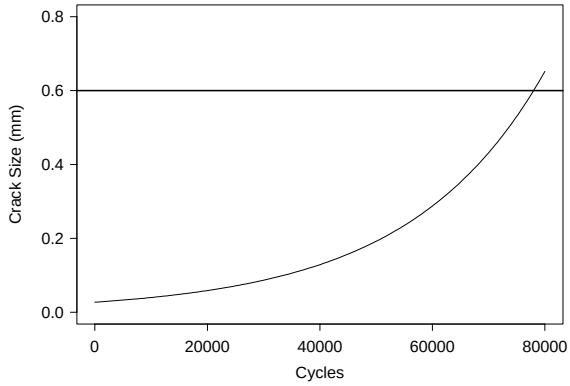
- Drive for efficiency could compromise reliability (safety).
- Level of "appropriate" or "tolerable risk" (10^{-5}, 10^{-6}, 10^{-7}, etc.)?
- Unwarranted assumptions about independence among input random variables could lead to serious in accuracies.
- Desired conclusions generally require extrapolation (in time, in stress, in experience)
- Best estimate or worst case bound? How quantified?
- Need to quantify uncertainty in input information.

A Generic Model for Degradation-Caused Failures

- Most failures traced to an underlying degradation process.
- Shape of degradation paths depends on physics/chemistry of the degradation process.
- Relationship between degradation and failure may be complicated.
- Failure time data useful; degradation data usually better.

Plot of Paris model for growth with fixed a_0, C, and m

$$\frac{da(t)}{dt} = C \times [\Delta K(a)]^m$$



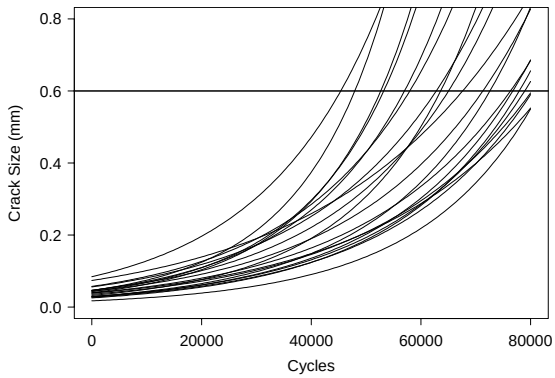
Models for Variation in Degradation and Failure Time

If all manufactured units were identical, operated at same time, under same conditions, in same environment, etc., then all units would fail at same time.

- Unit-to-unit variability in:
  - Initial level of degradation.
  - Material properties.
  - Component geometry, dimensions, alignment.
- Variable relationship between degradation and failure.
- Variable operating and environmental conditions (within and across units).

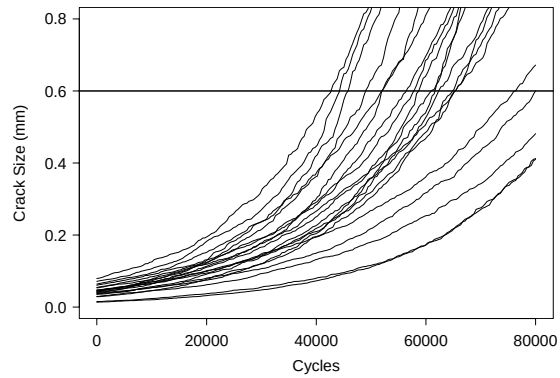
Plot of Paris model for growth of fatigue cracks (unit-to-unit variability in a_0, C, and m)

$$\frac{da(t)}{dt} = C \times [\Delta K(a)]^m$$

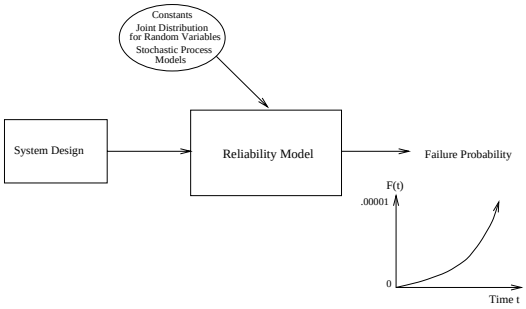


Growth of fatigue cracks (unit-to-unit variability in a_0, C, and m) and stochastic stress pattern

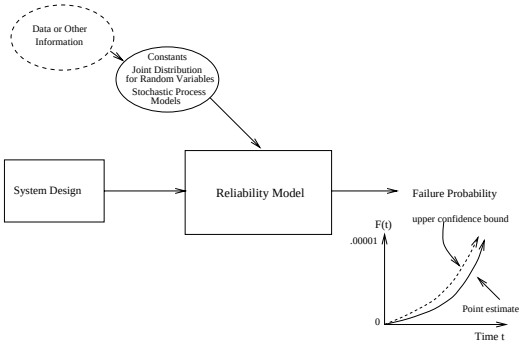
$$\frac{da(t)}{dt} = C \times [\Delta K(a)]^m$$



Probabilistic Design



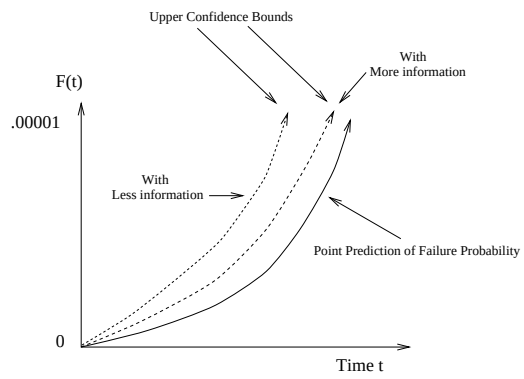
Probabilistic Design with Uncertainty in Inputs



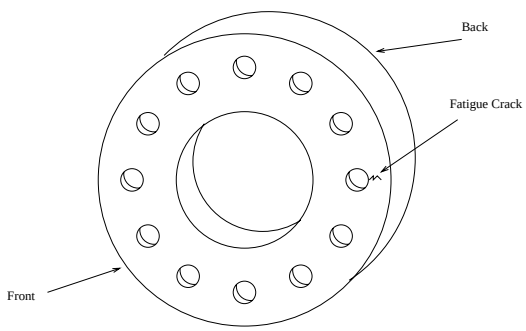
Separating Variability and Uncertainty

- Variability in product performance (e.g., life) is described by joint distributions of random variables (initial conditions) and stochastic processes (environmental variables).
- Variability defines the shape of the failure-time distribution.
- Uncertainty results from limited knowledge and limited data
  - Uncertainty in model form (especially important in extrapolation).
  - Uncertainty in model parameters (limited data).
- Uncertainty leads to uncertainty bounds (e.g., statistical confidence bounds quantify parameter uncertainty, but not model uncertainty). Can be reduced by collecting appropriate data (when possible).

Failure-time Distribution Estimate and Uncertainty Bounds



Example–Jet Engine Fan Disk



Random Variable/Stochastic Process Inputs to Jet Engine Fan Disk Reliability Model

- Internal
  - Crack growth parameters.
  - Material fracture toughness.
  - Material yield strength.
  - Geometrical and other manufacturing variations.
  - Material anomalies (ceramic or hard-alpha inclusions, voids, etc.).
- External
  - Temperature profile variation.
  - Stress profile variation.

Jet Engine Fan Disk Simplified Reliability Model

- Series system of independent "components" with non-identically distributed failure times. Resulting failure-time distribution for the disk is

$$F_T(t) = 1 - \prod_{i=1}^{s} (1 - F_i)$$

- Failure time of individual "components" modeled with a fatigue-fracture model using a Paris model for crack growth rate.

$$\frac{d\,a(t)}{d\,t} = C \times [\Delta K(a)]^m$$

C and m are "materials properties" that may depend on temperature, frequency, size, $\Delta K(a)$ depends on design parameters part dimensions and geometry, and stress profiles, etc.

Random Variables for the Simplified Jet Engine Fan Disk Reliability Model

Internal random variables:

- Paris parameters m, C.
- Time to initiation A_0 (or initial crack size T_I).

Incomplete knowledge about the distribution of m, C, and A_0.

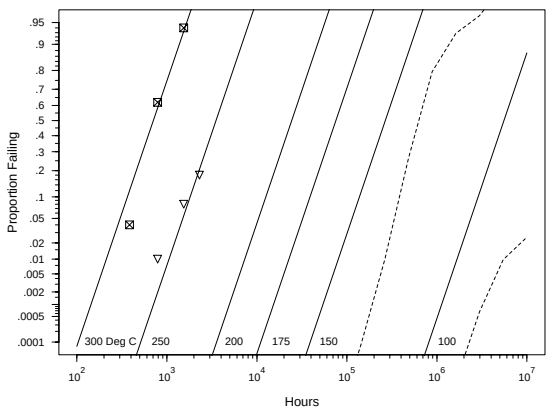
Probabilistic Design Options

When a reliability specification is not met (upper bound on failure probability is too large), possible options are:

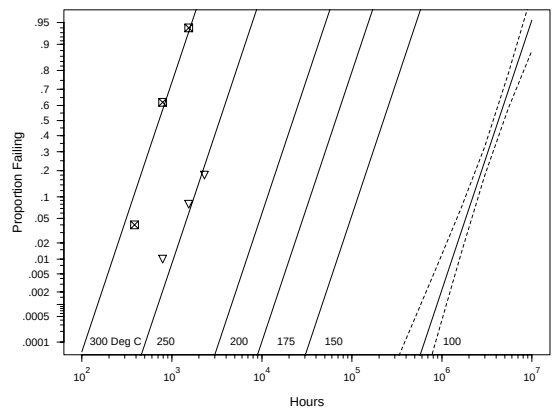
- Strengthen (or eliminate) the weak parts of the product design.
- Shorten the safe-life specification (replace earlier).
- Make product more robust to external forces (Taguchi ideas—highly touted statistical-engineering approach, but potential limited with an existing near-optimum design).
- Change the environment in which the weak parts operate (e.g., reduce temperature or stress).
- Experiment to reduce uncertainty (by obtaining better information about unknown inputs).

Operational Inputs to a Reliability Model

Lognormal Probability Plot Showing the Arrhenius-Lognormal Model ML Estimation Results a $n = 250$ ALT for a New-Technology IC Device



Lognormal Probability Plot Showing the Arrhenius-Lognormal Model ML Estimation Results for the New-Technology IC Device ALT with Given $E_a = .8$



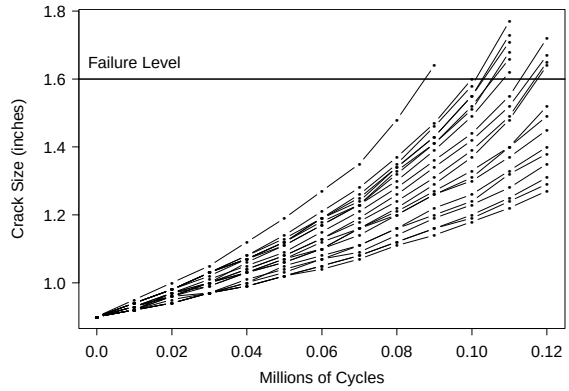
Choice of Experiments to Get Information about Fatigue/Fracture Failure Process

1. Fatigue crack growth experiments to get information on crack-growth parameters (e.g., Paris C and m).
2. Crack initiation experiments to get information on T_I or A_0.
3. Joint crack initiation/crack growth experiments to get information on C and m and T_I (or C, m and A_0).
4. Fatigue-fracture experiments to estimate S/N curves (or stress versus cycles-to-fracture curves).

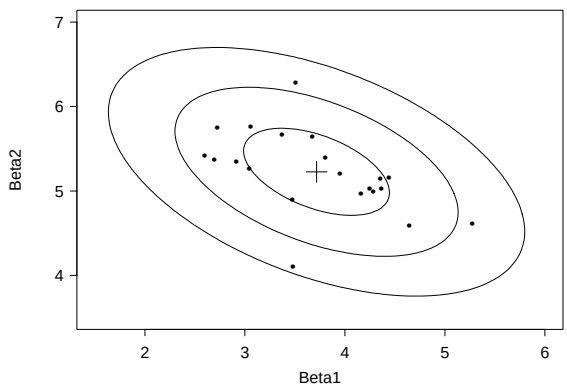
Different experiments have different per unit costs $\gamma_1, \ldots, \gamma_4$.

Estimation of variabilities is critical. Information needed as a function of stress and temperature. Need to quantify uncertainty due to limited sample size.

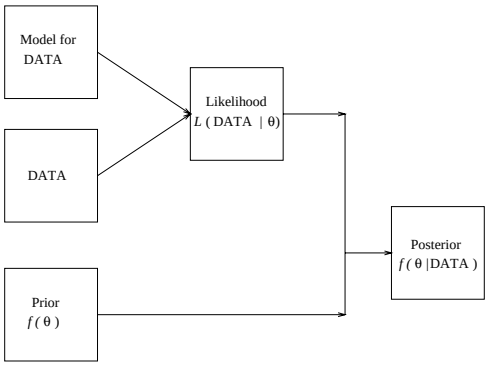
Fatigue Crack Size Observations for Alloy-A



Estimates $\widehat{m} = \widehat{\beta}_{1i}$ versus $\widehat{C} = \widehat{\beta}_{2i}$, $i = 1, \ldots, 21$ and Contours for the Fitted Bivariate Normal distribution



Bayes' Method for Inference



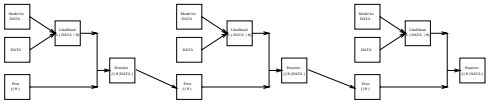
Hierarchical Bayes Model Describing Random Inputs for the Jet Engine Fan Disk Simplified Reliability Model

- Random inputs to reliability model: $\boldsymbol{\beta} = [\log(m), \log(C), \log(A_0)]$.
- $\boldsymbol{\beta} \sim \text{MVN}\left(\mu_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}\right)$.
- $\left(\mu_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}\right)$ is an unknown set of parameters $(3 + 6 = 9)$.
- Describe uncertainty with a "prior" distribution on $\left(\boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}\right)$.

$$\begin{aligned} \Sigma_{\boldsymbol{\beta}} &\sim \text{Inverse Wishart}_{\nu_0}\left(\boldsymbol{\Lambda}_0^{-1}\right) \\ \boldsymbol{\mu}_{\boldsymbol{\beta}} | \Sigma_{\boldsymbol{\beta}} &\sim \text{MVN}\left(\boldsymbol{\mu}_0, \Sigma_{\boldsymbol{\beta}}/\kappa_0\right). \end{aligned}$$

- $\nu_0, \boldsymbol{\Lambda}_0, \boldsymbol{\mu}_0, \kappa_0$ are hyperparameters $(1 + 6 + 3 + 1 = 11)$.
- These prior distributions are "conjugate" for independent observations from $\boldsymbol{\beta} = [\log(m), \log(C), \log(A_0)]$. Needed generalization for incomplete data has 18 parameters.

Sequential Updating Using Bayes' Theorem



Analysis

- With prior information only we get an estimate of failure-time CDF. The marginal distribution of $F(t_0)$ gives an upper confidence bound. Can repeat for range of t_0.
- Confidence bound may be too large to be useful due to uncertainty in inputs (especially with vague prior).
- Can conduct experiments to reduce uncertainty.
- Assess tradeoff between up front experimentation to reduce uncertainty and long-term cost of having to be conservative because of uncertainty.

Possible Optimization Problems for the Simple Example

Choose sample sizes $n_i \geq 0$, , $i = 1, 4$ to

- Use pre-posterior analysis allows an economic assessment of various experimental alternatives.

  Minimize E [Loss] with expectation over the experimental data and with respect to the available (prior) information and alternative decisions.

- Minimize $\text{E}\left[\widetilde{F}(t_e)\right]$

  Subject to: $\sum_{i=1}^{4} \gamma_i n_i \leq$ Experimental Budget.

Areas for Further Research

- Development of more and better physical/chemical models for failure-causing degradation.
- Computational methods for reliability models.
- Stochastic process degradation models needed for some applications (average or typical stress and average temperature often not good enough).
- Investigate methods of determining appropriate parameter transformation to obtain convenient distributional forms for life-driver random variables.
- Develop systematic methods for sensitivity analysis (e.g., to prior distribution inputs).
- Need to look at computationally efficient methods of doing posterior and pre-posterior analyses.
- Need to develop methods to quantify possible model error (in addition to the limited parameter information error).

Concluding Remarks

- Probabilistic design is or has moved into the operational stage for critical components in aircraft jet engines.
- Appropriate quantification of uncertainty is important for the application of probabilistic design methods.
- Protection against anomalies can dominate reliability computations in some applications.
- A formal Bayes structure provides a convenient, consistent, coherent framework within which we can combine all available information for the probability model and assess risk.
- When using Bayesian methods we need to protect against the use of wishful thinking for prior information, especially when the prior distribution dominates final answers.