1. For $t \in \mathbb{R}$

$$P[X_i \le t] = \Phi\left(\frac{t - \frac{1}{i}}{\frac{1}{i}}\right) = \Phi(it - 1)$$

So as $i \to \infty$, for $t \neq 0$

$$P[X_i \le t] \to \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$

and this agrees with the cdf of a unit point mass at 0. (The fact that $P[X_i \le 0]$ doesn't converge to 1, the value of that cdf at 0 is irrelevant, since 0 is not a continuity point of the limit cdf.) Yes, they converge in dsn

2. a) Obviously, if the marginal is itself normal since a linear combination of independent normals is normal $(S_n - ES_n)/\sqrt{\mathrm{Var}\, S_n}$ is <u>exactly</u> $N(0,1)$ for each n.

b) The corrected version of Problem 20 may be applied here. It suffices that

$$\frac{\mu_1^2}{\sum_{j=1}^{i} \mu_j^2} \longrightarrow 0$$

i.e. that $\sum_{j=1}^{\infty} \mu_j^2$ diverges

3. Let ϕ_n be the chf of X_n and ϕ be the chf of X. Then, for $t \in \mathbb{R}$,

$$E\, e^{itX_{N_n}} = E\,E\left[e^{itX_{N_n}} \mid N_n \right]$$

$$= E\, \phi_{N_n}(t)$$

But $|\phi_{N_n}(t)| \le 1$ and $N_n \to \infty$ a.s. implies that $\phi_{N_n}(t) \to \phi(t)$ a.s. So the BCT shows that

$$E\, \phi_{N_n}(t) \to \phi(t)$$

and the continuity theorem then implies that $X_{N_n} \xrightarrow{d} X$

4. For $t \in \mathbb{R}$, the function of x $\exp(tx)$ is bounded and cont's on $[a,b]$. Thus convergence in dsn of X_i to X implies that

$$E \exp(tX_i) \to E \exp(tX)$$

(This is 3) $\Rightarrow$ 1) in Theorem 2)

5. $Y = WX + (1-W)(-X)$

So if μ is the dsn of X and ν is the dsn of $-X$, the dsn of Y is

$$\tfrac{1}{2}\mu + \tfrac{1}{2}\nu$$

and
$$E \exp(itY) = \frac{1}{2}\int \exp(itz)\, d\mu(z) + \frac{1}{2}\int \exp(itz)\, d\nu(z)$$

$\int \exp(-itz)\, d\mu(z)$

$$= \frac{1}{2}\left(\frac{1}{1-it}\right) + \frac{1}{2}\left(\frac{1}{1+it}\right)$$

$$= \frac{1}{2} \frac{2}{(1-it)(1+it)} = \frac{1}{1+t^2}$$

This is real, so the dsn of Y is symmetric about 0.

6 a) The joint pmf for $(X_1, X_2, \dots X_n)$ on $\{0,1,2,\dots\}^n$ is

$$f(x) = \prod_{i=1}^{n} \left[\binom{m}{x_i} p^{x_i} (1-p)^{m-x_i} I[x_i \le m] \right]$$

$$= p^{\sum x_i} (1-p)^{nm - \sum x_i} I[\max x_i \le m]$$

$$\times \prod_{i=1}^{n} \binom{m}{x_i}$$

$$\frac{(m!)^n}{\prod_{i=1}^{n} x_i! \prod_{i=1}^{n} (m-x_i)!}$$

The factorization Theorem shows (as always) That The order statistics (or equivalently The empirical dsn) of the x_i's is sufficient. I'm 90% certain That these are minimal sufficient. If we are to have The functions of (m,p)

$$f(y|m,p) \propto f(x|m,p)$$

we must first have Them positive for The same set of m. This requires That The indicators

$$I[m \ge \max y_i] \text{ and } I[m \ge \max x_i]$$

are The same, i.e. $\max y_i = \max x_i$. Then, e.g. fixing p at $\frac{1}{2}$ we have to have the functions of m on $\{ \max x_i, \max x_i + 1, \dots \}$

$$\frac{1}{\prod_{i=1}^{n} (m - x_i)!} \propto \frac{1}{\prod_{i=1}^{n} (m - y_i)!}$$

i.e. $$\prod_{i=1}^{n} \Gamma(m - x_i + 1) \propto \prod_{i=1}^{n} \Gamma(m - y_i + 1)$$

and I'm 99% sure that one may proceed by induction on n to show that 2 functions

$$\prod_{i=1}^{n} \Gamma(m - a_i) \text{ and } \prod_{i=1}^{n} \Gamma(m - b_i)$$

can be proportional iff the a_i and b_i have the same dsns –

b) For the case of $n = 2$ the minimal sufficiency of the order statistic is easier to see and for $(s_1, s_2) \in \{0, 1, 2, \dots, m\}^2$

$$E_{m,p}\left[g(X_1, X_2) \mid S(X) = (s_1, s_2)\right]$$

$$= \frac{\sum_{x \text{ with } S(x) = (s_1, s_2)} g(x_1, x_2) P_{m,p}\left[X_1 = x_1 \text{ and } X_2 = x_2\right]}{P_{m,p}\left[S(X) = (s_1, s_2)\right]}$$

$$= \frac{\left(g(s_1, s_2) + g(s_2, s_1)\right) \binom{m}{s_1}\binom{m}{s_2} p^{s_1 + s_2} (1-p)^{2m - s_1 - s_2}}{2 \binom{m}{s_1}\binom{m}{s_2} p^{s_1 + s_2} (1-p)^{2m - s_1 - s_2}}$$

$$= \frac{1}{2} \left(g(s_1, s_2) + g(s_2, s_1) \right)$$

7. X is not complete. $h(X) = W = X_1$ is a nontrivial ancillary (and therefore, 1st order ancillary) statistic.

8. If $P[X=-1] = P[X=1] = \frac{1}{2}$ Then

$$Ee^{itX} = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}$$
$$= \cos t$$

Then for $X_1, X_2, \ldots, X_n$ iid with this dsn and $S_n = \sum_{i=1}^{n} X_i$

$$Ee^{it\frac{1}{\sqrt{n}}S_n} = E\,e^{i\left(\frac{t}{\sqrt{n}}\right)S_n}$$

$$\xrightarrow{\text{independence}} = \prod_{i=1}^{n} E\,e^{i\left(\frac{t}{\sqrt{n}}\right)X_i}$$

$$= \left(\cos\frac{t}{\sqrt{n}}\right)^n$$

But the CLT says $\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,1)$, so

$$\left(\cos\frac{t}{\sqrt{n}}\right)^n \longrightarrow e^{-\frac{t^2}{2}}$$) the std normal Chf