Stat 643 Exam 1 (Corrected)

March 5, 2007 **Prof. Vardeman**

Below are questions with a total of 10 parts. Each of these parts will be marked on a 10 point basis (100 points total possible). Write answers to as many of them as you have time for in 2 hours. Begin each new part of a problem on a new sheet of paper. Put the number and part of the problem at the top of each sheet. At the end of the exam period, staple the papers together in numerical order.

1. Suppose that $X_i \sim \mathrm{N}\left(\frac{1}{i}, \left(\frac{1}{i}\right)^2\right)$ (the standard deviation is $1/i$). Prove or disprove that these variables converge in distribution.

2. Suppose that $\{Y_i\}$ is a sequence of iid random variables with $\mathrm{E}\,Y_1 = 1$ and $\mathrm{Var}\,Y_1 = 1$. Let $W_i = \mu_i Y_i$ for each i and $S_n = \sum_{i=1}^{n} W_i$.

a) What is a marginal distribution for Y_1 under which $(S_n - \mathrm{E}S_n)/\sqrt{\mathrm{Var}\,S_n} \rightarrow^d \mathrm{N}(0,1)$ for any (not identically 0) sequence of reals $\{\mu_i\}$?

b) Give a sufficient condition on a *decreasing sequence of positive reals* $\{\mu_i\}$ such that for any marginal distribution for Y_1, $(S_n - \mathrm{E}S_n)/\sqrt{\mathrm{Var}\,S_n} \rightarrow^d \mathrm{N}(0,1)$. (Argue that your condition is indeed sufficient.)

3. Consider independent sequences of random variables on the same probability space, $\{X_n\}$ and $\{N_n\}$ with the properties that the N_n are positive integer valued,

$$X_n \rightarrow^d X \quad \text{and} \quad N_n \rightarrow \infty \ \text{a.s.}$$

Argue carefully that

$$X_{N_n} \rightarrow^d X$$

using conditioning and characteristic functions.

4. Suppose that $\{X_i\}$ is a sequence of random variables taking values in $[a,b]$ for real numbers $a < b$ and that

$$X_i \rightarrow^d X$$

Argue carefully that the moment generating functions for the X_i converge to that of X. That is, argue carefully that for $t \in \Re$

$$\mathrm{E}\exp(tX_i) \rightarrow \mathrm{E}\exp(tX)$$

5. Suppose that $X \sim \mathrm{Exp}(1)$ independent of $W \sim \mathrm{Ber}\left(\frac{1}{2}\right)$. Find the characteristic function of $Y = 2WX - X$ and use it to argue that Y has a distribution symmetric about 0. (The $\mathrm{Exp}(1)$ characteristic function is $\phi(t) = (1 - \imath t)^{-1}$.)

6. Suppose that $X_1, X_2, \ldots, X_n$ are iid Binomial (m, p) random variables, and neither $m \geq 1$ nor $p \in (0,1)$ is known.

a) Identify a minimal sufficient statistic for the parameter (m, p) and argue carefully that it is indeed minimal sufficient. (This is way harder than I intended. My original "solution" was wrong.)

b) For the case of $n = 2$, $\mathbf{T}$ your minimal sufficient statistic from a), and $g(x_1, x_2) : \{0, 1, 2, \ldots\}^2 \to \Re$, give an explicit formula for

$$\mathrm{E}\left[g(X_1, X_2) \,|\, \sigma(\mathbf{T}) \right]$$

and demonstrate that your prescription doesn't depend upon the parameter (m, p).

7. Suppose that $U, V,$ and W are independent random variables, $U \sim \mathrm{N}(\theta, 1)$, $V \sim \mathrm{N}(\theta, 4)$, and $W \sim \mathrm{Ber}\left(\frac{1}{2}\right)$. Define

$$Y = WU + (1 - W)V \quad \text{and} \quad \mathbf{X} = (W, Y)$$

Prove or disprove that $\mathbf{X}$ is complete and sufficient. (The intention was that the model under consideration is that for $\mathbf{X}$, not that for (U, V, W). But I marked either way.)

8. Use a probabilistic argument to identify $\lim_{n \to \infty} \left(\cos\left(\frac{t}{\sqrt{n}} \right) \right)^n$. (Argue carefully that your answer is correct.)