

Stat 643 Exam 2
(Corrected Version)
Spring 2010

I have neither given nor received unauthorized assistance on this exam.

Name Signed

Date

Name Printed

This exam is comprised of 10 equally weighted parts. Write answers for as many of those parts as you can in the 2 hours available for this exam. (You will surely NOT come close to finishing. Find things that you can do easily and completely.) Put your answers on the pages that follow, not on separate sheets.

1. Prove or give a counter-example to the following:

For a decision problem with finite parameter space $\Theta = \{1, 2, \dots, m\}$, if a prior distribution G is least favorable, it is unique. (That is, if $G' \neq G$ then G' is not least favorable.)

2. Consider a decision problem with finite parameter space $\Theta = \{1, 2, \dots, 9\}$, and action space consisting of two-element subsets of Θ , $\mathcal{A} = \{\{\theta, \theta'\} \mid \theta, \theta' \in \Theta, \theta \neq \theta'\}$. Then let

$$L(\theta, a) = I[\theta \notin a]$$

Suppose that a prior G on Θ and a likelihood $f_\theta(X)$ produce for each $x \in \mathcal{X}$ a posterior over Θ that we will symbolize as $G(\theta \mid x)$. Give a prescription (in as explicit a form as possible) for a Bayes decision rule versus the prior G .

3. Consider a decision-theoretic treatment of simple versus simple testing of $H_0 : X \sim f_0$ versus $H_1 : X \sim f_1$ with 0-1 loss, where

$$f_0(x) = I[0 < x < 1] \quad \text{and} \quad f_1(x) = 2xI[0 < x < 1]$$

are densities with respect to Lebesgue measure on \mathbb{R} . Find a minimax test and identify a least favorable prior for this problem.

4. In the testing context of problem 3, suppose that one observes only $Y \equiv X \cdot I[X < .8]$ (and not X itself). Find most powerful tests of every size $\alpha \in [0,1]$ based on Y .

5. Consider squared error loss estimation of $\theta > 0$ based on X_1 and X_2 that are iid $U(0, \theta)$. The method of moments estimator of θ is $\hat{\theta} = X_1 + X_2$. $M \equiv \max(X_1, X_2)$ is well-known to be sufficient for θ , and conditional distributions for (X_1, X_2) given M are uniform on the union of two line segments in \mathbb{R}^2

$$S_M = \{(x_1, x_2) \mid x_1 = M \text{ and } x_2 \in (0, M]\} \cup \{(x_1, x_2) \mid x_2 = M \text{ and } x_1 \in (0, M]\}$$

(Note that this means that marginally conditionally, X_1 is M with probability $\frac{1}{2}$ and with

probability $\frac{1}{2}$ is $U(0, M)$.) Rao-Blackwellize $\hat{\theta}$ to produce an explicit formula for an estimator.

What properties does this estimator have?

6. A certain realized log-likelihood $L_n(\theta_1, \theta_2)$ for the two real parameters θ_1 and θ_2 in a regular problem is maximized at $\theta_1 = 2.0$ and $\theta_2 = 3.0$, where the Hessian of the log-likelihood (the matrix of second partials) is

$$\begin{pmatrix} -225 & -25 \\ -25 & -100 \end{pmatrix}$$

What are approximate 95% confidence limits for θ_1 ? What is an approximate significance level for testing $H_0 : \theta_1 = 0$ and $\theta_2 = 0$? (You don't need to report a number, but tell me what tail probability of what distribution provides this.)

7. The interchange of orders of differentiation and integration is important to many of the arguments made in likelihood-based theory. For μ a sigma-finite measure on some space \mathcal{X} , suppose $g(x, \theta): \mathcal{X} \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable **and integrable** in x for every fixed θ and **continuously** differentiable in θ for every fixed x , and there exists some integrable $M(x) \geq 0$ such

$$\left| \frac{d}{d\theta} g(x, \theta) \right| \leq M(x) \quad \forall x \text{ and } \theta$$

Prove then that

$$\frac{d}{d\theta} \int g(x, \theta) d\mu(x) = \int \frac{d}{d\theta} g(x, \theta) d\mu(x)$$

(Hint: Begin by writing the left hand side as a limit of a difference quotient. A first order Taylor approximation becomes relevant.)

8. Suppose that X_1, X_2, \dots, X_n are iid on \mathbb{R} with marginal pdf $f_{\theta_1, \theta_2}(x)$ for real parameters θ_1 and θ_2 . Write

$$m_1(\theta_1, \theta_2) = E_{\theta_1, \theta_2} X \quad \text{and} \quad m_2(\theta_1, \theta_2) = E_{\theta_1, \theta_2} X^2$$

and suppose that the mapping from $\mathbf{m}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\mathbf{m}(\theta_1, \theta_2) = (m_1(\theta_1, \theta_2), m_2(\theta_1, \theta_2))$$

is one-to-one and differentiable, with (differentiable) inverse \mathbf{h} . Identify a consistent estimator of (θ_1, θ_2) and identify a condition sufficient to guarantee that your estimator is root- n consistent.

(Argue that your condition really is sufficient.)

9. Suppose that the density of X with respect to some sigma-finite measure μ is of 1-parameter exponential family form

$$f_{\eta}(x) = \mathcal{K}(\eta) \exp(\eta T(x))$$

for some real-valued function T . Under what conditions is the Cauchy-Schwarz inequality an equality? How then is it obvious that the Cramér-Rao inequality is achieved for the statistic $T(X)$? (Argue this not from a knowledge of what the C-R bound turns out to be, but from the condition that produces equality in Cauchy-Schwarz.)

10. Consider a "compound decision problem" as follows. $\Theta = \mathbf{a} = \{0,1\}^N$ (so both parameters and actions are N -vectors of 0's and 1's). Let

$$L(\theta, a) = \frac{1}{N} \sum_{i=1}^N I[a_i \neq \theta_i]$$

Suppose that for $\theta \in \Theta$, $\mathbf{X} = (X_1, X_2, \dots, X_N)$ has independent components, $X_i \sim N(\theta_i, 1)$. (This is N discriminations between $N(0,1)$ and $N(1,1)$ where loss is an overall error rate.)

Let $\pi : \{1, 2, \dots, N\} \rightarrow \{1, 2, \dots, N\}$ be a permutation of the integers 1 to N , and define for

$\mathbf{x} = (x_1, x_2, \dots, x_N)$ the transformation $\mathbb{R}^N \rightarrow \mathbb{R}^N$ $g_\pi(\mathbf{x}) = (x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(N)})$. Then

$$\mathcal{G} = \{g_\pi \mid \text{such that } \pi \text{ is a permutation}\}$$

is a group. Show that the problem is invariant and say explicitly as possible what it means for a non-randomized decision rule to be equivariant in this context.