Stat 643 Mid Term Exam
Fall 1996

October 22, 1996 Prof. Vardeman

1. Consider a model $\mathcal{P} = \{P_\theta\}$, where $\Theta = \{1, 2, 3\}$, $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and the distributions P_θ are specified in the table below.

	x_1	x_2	x_3	x_4
$\theta = 1$	.6	.1	.2	.1
$\theta = 2$	.3	.1	.2	.4
$\theta = 3$	0	.2	.4	.4

a) Find a minimal sufficient statistic $T : \mathcal{X} \to \mathcal{T}$, where $\mathcal{T}$ has 3 elements. Give the distribution of T in a tabular form similar to that used above for specifying the distribution of X.

Suppose that one is to make a decision about θ, where $\mathcal{A} = \Theta = \{1, 2, 3\}$ and $L(\theta, a) = I[a \neq \theta]$.

b) Consider the behavioral decision rule ϕ defined by

$$\begin{array}{lll} \phi_{x_1}(\{1\}) = .8 & \phi_{x_1}(\{2\}) = .2 & \phi_{x_1}(\{3\}) = 0 \\ \phi_{x_2}(\{1\}) = .6 & \phi_{x_2}(\{2\}) = .3 & \phi_{x_2}(\{3\}) = .1 \\ \phi_{x_3}(\{1\}) = .3 & \phi_{x_3}(\{2\}) = .6 & \phi_{x_3}(\{3\}) = .1 \\ \phi_{x_4}(\{1\}) = 0 & \phi_{x_4}(\{2\}) = .2 & \phi_{x_4}(\{3\}) = .8 \end{array}$$

Find a rule that is a function of your sufficient statistic from a) and is risk equivalent to ϕ.

c) Argue directly from the definition of admissibility that $\delta_1(x) \doteq 1$ is admissible.

d) Show that

$$\delta_2(x) = \begin{cases} 1 & \text{if } x = x_1 \text{ or } x = x_2 \\ 3 & \text{if } x = x_3 \text{ or } x = x_4 \end{cases}$$

is Bayes versus a prior G that places mass $\frac{2}{3}$ on $\theta = 1$ and mass $\frac{1}{3}$ on $\theta = 3$.

e) Show that δ_2 defined in d) is admissible.

f) Consider a randomized rule ψ in this decision problem defined by $\psi(\{\delta_1\}) = \frac{1}{2}$ and $\psi(\{\delta_2\}) = \frac{1}{2}$. Find a behavioral decision rule ϕ' that is risk equivalent to ψ.

2. Consider the family of distributions $\mathcal{P} = \{P_{\boldsymbol{\theta}}\}$ on $\mathcal{X} = [0, \infty)$ indexed by the parameter $\boldsymbol{\theta}=(\theta_1, \theta_2) \in \Theta \subset \mathcal{R}^2$, dominated by the σ-finite measure $\mu = \Delta + \lambda$ for Δ a unit point mass at 0 and λ Lebesgue measure on $\mathcal{X}$, where

$$f_{\boldsymbol{\theta}}(x) = \frac{dP_{\boldsymbol{\theta}}}{d\mu}(x) \propto \exp(\theta_1 I[x = 0] + \theta_2 x) \quad .$$

a) What are the natural parameter space and "normalizing constant" $K(\boldsymbol{\theta})$ for this family of distributions?

b) Use $K(\boldsymbol{\theta})$ to find the moment generating function for $X \sim P_{\boldsymbol{\theta}}$, $\mathrm{E}_{\boldsymbol{\theta}} \exp sX$.

c) Identify the UMVUE for $\gamma(\boldsymbol{\theta}) = P_{\boldsymbol{\theta}}[X_1 = 0]$ based on $X_1, X_2, ..., X_n$ iid $P_{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}$ in the natural parameter space. Argue very carefully that your estimator is UMVU.

3. Consider squared error loss estimation of the binomial parameter "p."

a) Write out (and to the extent possible simplify) the risk function for a linear estimator of p, $\delta'(x) = Ax + B$.

Using the result from a) it is possible to show that $\delta(x) = \frac{x+\frac{\sqrt{n}}{2}}{n+\sqrt{n}}$ has constant risk. (Don't bother to show this here.) It is also the case that if conditional on p, $X \sim$ Binomial (n, p) and $p \sim$ Beta (α, β), E$[p|X = x] = \frac{\alpha+x}{\alpha+\beta+n}$. (Again, you need not show this here.)

b) Argue carefully that δ is an admissible estimator of p.

c) Take the result in b) as given and prove that δ is the **unique** minimax rule for this estimation problem.