Stat 643
Solution Key to Homework Assignment 8
Apr 24, 2007

Q64

Suppose $\Theta = \{\theta_1, \ldots, \theta_k\}$, $\mathcal{A} = \mathcal{S}$ is a subset of $[0, +\infty)^k$ and

$$P_\theta(X = 0) = 1, \qquad \forall \theta \in \Theta$$

Let the loss function be defined by

$$L(\theta_i, a) = a_i, \qquad \forall 1 \leqslant i \leqslant k, \quad a \in \mathcal{A}$$

Then for any $y \in \mathcal{S}$, let $\delta(x) = (y_1, \ldots, y_k)$. Thus

$$R(\theta_i, \delta) = \int_{\mathcal{X}} L(\theta_i, \delta(x)) \, \mathrm{d}P_{\theta_i}(x) = y_i$$

i.e., the risk vector

$$\big(R(\theta_1, \delta), \ldots, R(\theta_1, \delta)\big) = (y_1, \ldots, y_k) = y$$

Q65

(a) There are 4 nonrandomized rules

$$\begin{aligned} &\delta_1(0) = 1 = \delta_1(1) \\ &\delta_2(0) = 1, \quad \delta_2(1) = 2 \\ &\delta_3(0) = 2, \quad \delta_3(1) = 1 \\ &\delta_4(0) = 2 = \delta_4(1) \end{aligned}$$

with corresponding risk vectors

$$\begin{aligned} \big(R(1, \delta_1), R(2, \delta_1)\big) &= (0, 1) \\ \big(R(1, \delta_2), R(2, \delta_2)\big) &= (0.25, 0.5) \\ \big(R(1, \delta_3), R(2, \delta_3)\big) &= (0.75, 0.5) \\ \big(R(1, \delta_4), R(2, \delta_4)\big) &= (1, 0) \end{aligned}$$

(c) The two segments accentuated represent the admissible risk vectors for this problem. By Thm 142, this set of admissible risk vectors is also a minimal complete class.

(d) Study the Bayes risk is to draw line of the form: $py_1 + (1-p)y_2 = c$, which has a slope of $-\frac{p}{1-p}$. After some basic geometry you can get the corresponding sets of risk vectors:

$$\begin{aligned}
&\{(0,1)\}, && \text{for } \frac{2}{3} < p \leqslant 1 \\
&\{(y_1,y_2) : 2y_1 + y_2 - 1 = 0, y_1 \in [0, 0.25]\}, && \text{for } p = \frac{2}{3} \\
&\{(0.25, 0.5)\}, && \text{for } \frac{2}{5} < p < \frac{2}{3} \\
&\{(y_1,y_2) : 2y_1 + 3y_2 - 2 = 0, y_1 \in [0.25, 1]\}, && \text{for } p = \frac{2}{5} \\
&\{(1,0)\}, && \text{for } 0 \leqslant p < \frac{2}{5}
\end{aligned}$$

For priors with $p = \frac{2}{3}$ and $p = \frac{2}{5}$, there are more than one Bayes rule.

(e) For a prior $g = (1/2, 1/2)$

$$\begin{aligned}
P[X = 0 \text{ and } \theta = 1] &= 1/2 \times 3/4 = 3/8 \\
P[X = 1 \text{ and } \theta = 1] &= 1/2 \times 1/4 = 1/8 \\
P[X = 0 \text{ and } \theta = 2] &= 1/2 \times 1/2 = 1/4 \\
P[X = 1 \text{ and } \theta = 2] &= 1/2 \times 1/2 = 1/4
\end{aligned}$$

Then $P[\theta = 1|X = 0] = \frac{3/8}{3/8+1/4} = 3/5$, so $P[\theta = 2|X = 0] = 2/5$.
Also $P[\theta = 1|X = 1] = \frac{1/8}{1/8+1/4} = 1/3$, so $P[\theta = 2|X = 1] = 2/3$.

Therefore, for $L(\theta, a) = I[\theta \neq a]$ we will choose 1 when $X = 0$ and 2 when $X = 1$ to 'minimize the conditional expected loss', this is our δ_2 which has risk vector (0.25,0.5). This is Bayes and is the same result as we get in (d).

Q66

(a) The risk vector of ϕ_x is given be

$$R(1, \phi_x) = \frac{1}{2} P_1(X_1 = 1) = \frac{1}{8}, \quad R(2, \phi_x) = P_2(X_1 = 0) + \frac{1}{2} P_2(X_1 = 1) = \frac{3}{4}$$

Let ϕ'_x be a rule defined by

$$\begin{array}{ll} \phi'_x(\{1\}) = 1, & \text{if } x_1 + x_2 \leqslant 1 \\ \phi'_x(\{2\}) = 1, & \text{if } x_1 + x_2 = 2 \end{array}$$

Then the risk vector for ϕ'_x is given by

$$\begin{aligned} R(1, \phi_x) &= P_1(X_1 = 1, X_2 = 1) = \frac{1}{16} < R(1, \phi_x) \\ R(2, \phi_x) &= 1 - P_2(X_1 = 1, X_2 = 1) = \frac{3}{4} = R(2, \phi_x) \end{aligned}$$

Therefore, ϕ'_x is a better rule. ϕ_x is inadmissible.

(b) Note that in this problem $T = X_1 + X_2$ is sufficient.

By Result 144, we can obtain a rule ϕ_T with the same risk as ϕ_x based on T as follows

(i) $T = 0$. In this case, we know that $X_1 = 0,\ a.s.$ and hence

$$\phi_T(\{1\}) = 1$$

(ii) $T = 1$. In this case, we know that $(X_1, X_2)|T$ is uniformly distributed on (0,1) and (1,0). Thus

$$\phi_T(\{1\}) = \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}, \qquad \phi_T(\{2\}) = \frac{1}{4}$$

(iii) $T = 2$. In this case, we know that $X_1 = 1,\ a.s.$ and thus

$$\phi_T(\{1\}) = \frac{1}{2}, \qquad \phi_T(\{2\}) = \frac{1}{2}$$

Q67

(a) $\forall p \in (0, 1)$, the risk function for δ_1 is

$$\begin{aligned} R(p, \delta_1) &= \int_{\mathcal{X}} \left(\frac{x}{n} - p\right)^2 \, \mathrm{d}P_p(x) \\ &= \frac{1}{n^2} \mathrm{Var} X \\ &= \frac{p(1-p)}{n} \end{aligned}$$

$\forall p \in (0,1)$, the risk function for δ_2 is

$$\begin{aligned} R(p,\delta_2) &= \int_{\mathcal{X}} \left(\frac{1}{2}\left(\frac{x}{n} + \frac{1}{2} \right) - p \right)^2 \ \mathrm{d}P_p(x) \\ &= \frac{1}{4n^2}\mathrm{Var}X + \left(\frac{1}{4} - \frac{p}{2} \right)^2 \\ &= \frac{p(1-p)}{4n} - \frac{p(1-p)}{4} + \frac{1}{16} \end{aligned}$$

$\forall p \in (0,1)$, the risk function for ψ is

$$\begin{aligned} R(p,\psi) &= \int_{\mathcal{D}} R(p,\delta) \ \mathrm{d}\psi(\delta) \\ &= \frac{1}{2}R(p,\delta_1) + \frac{1}{2}R(p,\delta_2) \\ &= \frac{5p(1-p)}{8n} - \frac{p(1-p)}{8} + \frac{1}{32} \end{aligned}$$

(b) Define a behavioral rule ϕ as follows

$$\phi\left(\left\{ \frac{x}{n} \right\} \right) = \frac{1}{2} = \phi\left(\left\{ \frac{1}{2}\left(\frac{x}{n} + \frac{1}{2} \right) \right\} \right), \quad \forall x$$

Then

$$\begin{aligned} R(p,\phi) &= \int_{\mathcal{X}} \int_{\mathcal{A}} L(p,a) \ \mathrm{d}\phi(a) \ \mathrm{d}P_p(x) \\ &= \frac{1}{2}R(p,\delta_1) + \frac{1}{2}R(p,\delta_2) \\ &= R(p,\psi) \end{aligned}$$

i.e., ϕ is risk equivalent to ψ.

(c) By Lemma 131 and the loss is trictly convex in a dsn on $[0,1]$, a better nonrandomized rule δ can be obtained as follows

$$\begin{aligned} \delta(x) &= \int_{\mathcal{A}} a \ \mathrm{d}\phi(a) \\ &= \frac{1}{2}\frac{x}{n} + \frac{1}{2}\cdot\frac{1}{2}\left(\frac{x}{n} + \frac{1}{2} \right) \\ &= \frac{3x}{4n} + \frac{1}{8} \end{aligned}$$

Q68

Suppose ϕ is not admissible when the parameter space is Θ. Then there exists ϕ' such that

$$R(\theta, \phi') \leqslant R(\theta, \phi), \qquad \forall \theta \in \Theta$$

and for some $\theta' = (\theta'_1, \theta'_2) \in \Theta$,

$$R(\theta', \phi') < R(\theta', \phi) \qquad (*)$$

But for all $\theta_2 \in \Theta_2$, ϕ is admissible when the parameter space is $\Theta_1 \times \theta_2$, i.e. no one beats me there. In particular, for θ'_2, we have that for all ϕ'',

$$R\big((\theta_1, \theta'_2), \phi\big) \leqslant R\big((\theta_1, \theta'_2), \phi''\big), \qquad \forall \theta_1 \in \Theta_1$$

But this contradicts $(*)$ when $\phi'' = \phi'$ and $\theta_1 = \theta'_1$. Therefore, ϕ is also admissible when the parameter space is Θ.

Q69

Since $w(\theta) > 0$, $\forall \theta$, we have

$$\begin{aligned}
& \phi \text{ is admissible with loss function } L(\theta, a) \\
\iff\quad & R_L(\theta, \phi) \leqslant R_L(\theta, \phi'), \qquad \forall \phi' \\
\iff\quad & \int_{\mathcal{A}} L(\theta, a)\ \mathrm{d}\phi(a) \leqslant \int_{\mathcal{A}} L(\theta, a)\ \mathrm{d}\phi'(a), \qquad \forall \phi' \\
\iff\quad & \int_{\mathcal{A}} w(\theta) L(\theta, a)\ \mathrm{d}\phi(a) \leqslant \int_{\mathcal{A}} w(\theta) L(\theta, a)\ \mathrm{d}\phi'(a), \qquad \forall \phi' \\
\iff\quad & R_{wL}(\theta, \phi) \leqslant R_{wL}(\theta, \phi'), \qquad \forall \phi' \\
\iff\quad & \phi \text{ is admissible with loss function } w(\theta) L(\theta, a)
\end{aligned}$$

Q69

(a)
The risk for $\bar{X}$ under squared error loss is $\mathrm{Var}(\bar{X}) = \frac{\theta(1-\theta)}{n}$.

(i) For n even, define

$$f_n(\theta) = \sum_{i=0}^{\frac{n}{2}-1} \binom{n}{i} \theta^i (1-\theta)^{n-i}$$

and for n odd, define

$$f_n(\theta) = \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{i} \theta^i (1-\theta)^{n-i}$$

Then

$$P_\theta(T_0(X)=0)=f_n(\theta), \qquad P_\theta(T_0(X)=1)=f_n(1-\theta)$$

and

$$P_\theta\left(T_0(X)=\frac{1}{2}\right)=\begin{cases}\binom{n}{\frac{n}{2}}\theta^{\frac{n}{2}}(1-\theta)^{\frac{n}{2}}, & \text{if } n \text{ is even}\\ 0, & \text{if } n \text{ is odd}\end{cases}$$

Hence, the risk for $T_0(X)$ is

$$R(\theta,T_0(X))=\begin{cases}\theta^2 f(\theta)+(\theta-1)^2 f(1-\theta)+\left(\theta-\frac{1}{2}\right)^2\binom{n}{\frac{n}{2}}\theta^{\frac{n}{2}}(1-\theta)^{\frac{n}{2}}, & \text{if } n \text{ is even}\\ \theta^2 f(\theta)+(\theta-1)^2 f(1-\theta), & \text{if } n \text{ is odd}\end{cases}$$

(b) The risk for $T_1(X)$ is given by

$$R(\theta,T_1(X))=\frac{1}{2}R(\theta,\bar{X})+\frac{1}{2}R(\theta,T_0(X))=\frac{\theta(1-\theta)}{2n}+\frac{1}{2}R(\theta,T_0(X))$$

where $R(\theta,T_1(X))$ is given in (a).

The risk for $T_2(X)$ is given by

$$\begin{aligned}R(\theta,T_2(X)) &= \mathrm{E}_\theta\left[\bar{X}L(\theta,\bar{X})+(1-\bar{X})L\left(\theta,\frac{1}{2}\right)\right]\\ &= \mathrm{E}_\theta\left[\bar{X}(\theta-\bar{X})^2+(1-\bar{X})\left(\theta-\frac{1}{2}\right)^2\right]\\ &= \mathrm{E}_\theta\left[(\bar{X})^3-2\theta(\bar{X})^2+\theta\bar{X}\right]+(1-\theta)\left(\theta-\frac{1}{2}\right)^2\\ &= \frac{1}{n^2}[(n-1)(n-2)\theta^3+2(n-1)\theta^2+\theta]-\frac{1}{n}[(n-2)\theta^3+2\theta^2]\\ &\quad +(1-\theta)\left(\theta-\frac{1}{2}\right)^2\end{aligned}$$

Q69

(i) Thm 2.5 can't improve anything on $T_0(X)$, it is not a random rule.

(ii) A rule δ_1 defined as follows is better than $T_1(X)$,

$$\delta_1(x)=\int_{\mathcal{A}} a \,\mathrm{d}\phi_{T_1}(a)=\frac{1}{2}\bar{X}+\frac{1}{2}T_0(X)$$

(iii) δ_2 defined as follows is better than $T_2(X)$,

$$\delta_2(x)=\int_{\mathcal{A}} a \,\mathrm{d}\phi_{T_2}(a)=(\bar{X})^2+\frac{1}{2}(1-\bar{X})$$