Stat 643
Solution Key to Homework Assignment 3
Feb 19, 2007

Q19.

If we put the mass out too fast, say in 2^n rate, then it won't converge since the last one dominate. If it stay at constant, then it will definitely converge. Where is the critical value in the functional space? Polynomial? Could be. Say X_j are independent r.v. with $P[X_j = \pm j^\alpha] = \frac{1}{2}$
$B_i^2 = \sum_{j=1}^i j^{2\alpha} = O(i^{2\alpha+1})$
Let's check Liapounov's condition, $\forall \delta > 0$

$$\begin{aligned}\frac{\sum_{j=1}^i E|X_j|^{2+\delta}}{B_i^{2+\delta}} &= \frac{\sum_{j=1}^i j^{\alpha(2+\delta)}}{i^{\frac{1}{2}(2\alpha+1)(2+\delta)}} \\ &= \frac{O(i^{2\alpha+\delta\alpha+1})}{O(i^{2\alpha+\delta\alpha+1+\frac{\delta}{2}})} \\ &\to 0 \quad \text{as } i \to \infty\end{aligned}$$

So S_i/B_i will have a normal limit as long as X_j increase in polynomial. In this problem, X_j increase in square root rate and take 2 as δ will easily satisfy Liapounov's condition.

Q20.

We have to use Lindeberg's condition since X's may not have moments beyond the 2nd moment. WOLOG, assume $\mu = 0$, $B_i^2 = \sigma_i^2 = \sigma^2 \sum_{j=1}^i z_{ij}^2$. Lindeberg's condition, $\forall \epsilon > 0$,

$$\begin{aligned}\frac{1}{\sigma_i^2}\sum_{j=1}^i \mathrm{E}[z_{ij}^2 X_j^2 I(|z_{ij}X_j| > \epsilon\sigma_i)] &= \frac{1}{\sigma_i^2}\sum_{j=1}^i z_{ij}^2 \mathrm{E}[X_j^2 I(X_j^2 > \frac{\epsilon^2\sigma_i^2}{z_{ij}^2})] \\ &\le \frac{1}{\sigma^2}\mathrm{E}[X_1^2 I(X_1^2 > \frac{\epsilon^2\sigma_i^2}{\max z_{ij}^2})]\end{aligned}$$

We have to assume $\frac{\sigma_i^2}{\max z_{ij}^2} \to \infty \Leftrightarrow \frac{\max z_{ij}^2}{\sum_{j=1}^i z_{ij}^2} \to 0$, i.e. no single z_{ij} dominate. Then the above $\to 0$ by DCT and Lindeberg's condition satisfied.

Q21.

$P[R_i = 1] = P\{Z_i > Z_j, \forall j < i\} = \frac{1}{i}$ for Z_i are iid continuous, $Z_1 \dots Z_i$ have equal chance to be the biggest one.
Independence: $j < i, P[R_i = 1, R_j = 1] = P[R_i = 1]P[R_j = 1|R_i = 1] = P[R_i = 1]P[R_j = 1]$. This is because knowing $R_i = 1$ has nothing to contribute to the event whether Z_j will be the biggest among first j r.v.s.
CLT can be established by either Liapounov's or Lindeberg's condition.

Q22.

Let r.v X has the density $f(x) = \frac{C}{x^3(\log x)^2} I(x \geq e)$, where C is the normalizing constant for $f(x)$ to be a p.d.f. $\forall \delta > 0$

$$
\begin{aligned}
\mathrm{E}X^2 &= \int_e^\infty \frac{Cx^2}{x^3(\log x)^2} dx \\
&= C \int_e^\infty \frac{1}{x(\log x)^2} dx \\
&= C \int_1^\infty \frac{1}{y^2} dy \\
&= C
\end{aligned}
$$

$\forall \delta > 0$

$$
\begin{aligned}
\mathrm{E}X^{2+\delta} &= \int_e^\infty \frac{Cx^{2+\delta}}{x^3(\log x)^2} dx \\
&= C \int_1^\infty \frac{e^{\delta y}}{y^2} dy \\
&\rightarrow \infty \qquad \text{for } \frac{e^{\delta y}}{y^2} \rightarrow \infty \text{ as } y \rightarrow \infty
\end{aligned}
$$

This means Liapounov's condition fails, while 2nd moment exists implies Lindeberg's condition.

Q23

Let X_i's be independent r.v's s.t.
$P(X_i) = \pm 1) = \frac{1}{2} - \frac{1}{2^{i+1}}, P(X_i = \pm 2^{j^2}) = \frac{1}{2^{i+j+1}}, \forall i, j \in N$

then $\forall \alpha > 0$,

$$\begin{aligned} \mathrm{E}|X_i|^\alpha &= (1-\frac{1}{2^i}) + \sum_{j=1}^{\infty} (2^{j^2})^\alpha \frac{1}{2^{i+j+1}} \\ &= 1-\frac{1}{2^i} + \frac{1}{2^{i+1}} \sum_{j=1}^{\infty} 2^{\alpha j^2 - j} \\ &= \infty \end{aligned}$$

for $2^{\alpha j^2 - j} \to \infty$ as $j \to \infty$
Then define $X'_i = X_i I(|X_i| = 1)$ and follow the suit in 2nd example in Dr. Vardeman's Feb 1st's note.
Note: The only thing differ is that the definition of K_n, now is that $K_n = \max\{k : \sum_{i=1}^k |X_i - X'_i| \le \epsilon\sqrt{n}\}$.

Q25

Check Liapounov's condition: take $\delta = 1$, $\mathrm{E}|X_{nj}|^3 = \frac{j^3}{4n^{\frac{9}{2}}}$, $B_n^3 = (\frac{\sum_{j=1}^n j^2/3}{n^3})^{\frac{3}{2}} = O(1)$
$\frac{\sum_{j=1}^n E|X_{nj}|^3}{B_n^3} = \frac{O(n^4)}{n^{\frac{9}{2}}} \to 0$
Then Liapounov's $\Longrightarrow$ Lindeberg's $\Longrightarrow$ Feller's. The last part is by Thm 7.2.1 of Chung.

Q28

Let $\{Y_j\}$ be independent r.v.'s with
$P[Y_j = \pm 1] = \frac{1}{2j}$ and $P[Y_j = 0] = 1 - \frac{1}{j}, \forall j \in N$.
Let $X_j = Y_j + Y_{j+1}$, Then $\sigma_n^2 \simeq 4\sum_{j=1}^n \frac{1}{j} = O(\log(n)) \to \infty$ and $\sigma_n / n^{1/3} \to 0$.
Let $S'_n = \sum_{j=1}^n Y_j$, and $\sigma_n^{'2}$ be the variance of S'_n.

Check Liapounov's condition for Y_j's: $\forall \delta > 0$

$$\begin{aligned} \frac{\sum_{j=1}^n E|Y_j|^{2+\delta}}{\sigma_n^{'(2+\delta)}} &= \frac{\sum_{j=1}^n \frac{1}{j}}{\sigma_n^{'(2+\delta)}} \\ &= \frac{O(\log(n))}{O(\log(n)^{1+\frac{\delta}{2}})} \\ &\to 0 \qquad \text{as } n \to \infty \end{aligned}$$

So we have CLT for the Y_j's, then $S_n = \sum_{j=1}^n X_j$ is roughly 2 times S'_n, i.e. $\frac{S_n - 2S'_n}{\sigma_n} = \frac{Y_{n+1} - Y_1}{\sigma_n} \to 0$ $w.p.1$.
And σ_n of X_j's is roughly 2 times that of Y_j's, i.e. $\frac{\sigma_n}{2\sigma'_n} \to 1$ $w.p.1$.
So a CLT for the 2-dependent X_j's is straightforward by Slutsky's.