# Stat 643
# Solution Key to Homework Assignment 1
# Jan 26, 2007

This is only the sketch of the proof, you will need to fill in the details and show/convince me there is sound reasoning from step to step. Good luck.

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**Q1.**

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Draw the $F_i(x)$ and $F(x)$ as the limit when $i \to \infty$(See Fig 1 at the end of solution).
Let measure $\mu$ be that $\mu((-\infty, x]) = \frac{1}{6}\lambda((-\infty, x]) + \frac{1}{3}I(x > \frac{1}{2}) + \frac{1}{3}I(x > 1.5)$
Argue that $\forall x \in \Re - \{0.5, 1.5\}, F_i(x) = \mu_i((-\infty, x]) \to \mu((-\infty, x]) = F(x)$

Follow the construction of Skorohod's Thm, let us define the inverse of c.d.f as

$$\begin{aligned} h_i(\omega) &= \sup\{t : \mu_i(-\infty, t] < \omega\} \\ h(\omega) &= \sup\{t : \mu(-\infty, t] < \omega\} \end{aligned}$$

where $h_i$ and $h$ are r.v on $((0, 1), B((0, 1)), m)$. The hardest part is $h_i$, where for $i \geq 4$

$$h_i(\omega) = \begin{cases} 6\omega & \omega \in (0, \frac{1}{12} + \frac{1}{6i}) \\ \frac{1}{2} + \frac{1}{i} & \omega \in [\frac{1}{12} + \frac{1}{6i}, \frac{5}{12} + \frac{1}{6i}) \\ 6\omega - 2 & \omega \in [\frac{5}{12} + \frac{1}{6i}, \frac{1}{2}] \\ \frac{3\omega - 1.5}{i} + 1.5 & \omega \in (\frac{1}{2}, \frac{5}{6}] \\ 6\omega - 3 & \omega \in (\frac{5}{6}, 1) \end{cases}$$

Note: the point of $\frac{5}{12} + \frac{1}{6i}$ could be open or closed.

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**Q2. Rosenthal 10.1.2**

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Enough to show $\forall x \in C(F), F_i(x) \to F(x)$
f is continuous on a bounded(bdd) interval, by property of Riemann integral we have
$\lim_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i} f(\frac{j}{i}) = \int_0^1 f(t)dt = 1$
$\lim_{i\to\infty} \frac{1}{i} \sum_{\frac{j}{i} \leq x} f(\frac{j}{i}) = \int_0^x f(t)dt$

$$\begin{aligned} \lim_{i\to\infty} F_i(x) &= \lim_{i\to\infty} \mu_i((-\infty, x]) \\ &= \lim_{i\to\infty} \frac{\sum_{\frac{j}{i} \leq x} f(\frac{j}{i})}{\sum_{j=1}^{i} f(\frac{j}{i})} \\ &= \frac{\int_0^x f(t)dt}{\int_0^1 f(t)dt} \\ &= \mu((-\infty, x]) = F(x) \end{aligned}$$

Let $Y(\omega) = "F^{-1}"(\omega) = \sup\{t : \mu((-\infty, t]) < \omega\}$
The question ask us to "explicitly" construct $Y_i$, so it goes:

$$\begin{aligned} Y_i(\omega) = "F_i^{-1}"(\omega) &= \sup\{t : \mu_i((-\infty, t]) < \omega\} \\ &= \frac{J}{i} \quad \text{if} \quad \frac{\sum_{j=1}^{J-1} f(x_j)}{\sum_{j=1}^{i} f(x_j)} < \omega \leq \frac{\sum_{j=1}^{J} f(x_j)}{\sum_{j=1}^{i} f(x_j)} \end{aligned}$$

Then by Skorohod's Thm $Y_i \to_d Y$

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**Q3. Rosenthal 10.1.4**

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The measure $\rho$ in Q1 should give one some hint on how to solve this one. To be a.c. w.r.t Lebesque measure on $R$, all we need is to eliminate "jump" in the c.d.f. One construction is like this:
let $\rho_{ij}$ = Lebesgue measure on $[j - \frac{1}{i}, j]$
$\mu_i = \sum_{j=1}^{\infty} a_i \, i \, \rho_{ij}$
See Fig 2 at the end for some illumination.

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**Q4. Rosenthal 10.1.6**

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An application of BCT

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**Q6. Convergence of Type**

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Let $Y_i = \frac{X_i - a_i}{b_i}$, $Z_i = \frac{X_i - \alpha_i}{\beta_i}$, then $Y_i \to_d X$
$\forall t \in C(F_X)$

$$\begin{aligned} F_{Z_i}(t) &= F_{Y_i}(\frac{\beta_i}{b_i} t + \frac{\alpha_i - a_i}{b_i}) && (1) \\ &\to_{i\to\infty} F_X(\frac{\beta_i}{b_i} t + \frac{\alpha_i - a_i}{b_i}) && (2) \\ &\to_{i\to\infty} F_X(t) && (3) \end{aligned}$$

(1) is simple arithmetic
(2) is because $C(F_X)$ is dense in $R$ so $\forall t \in C(F_X)$, you can find i large enough so that $\frac{\beta_i}{b_i} t + \frac{\alpha_i - a_i}{b_i}$ is in near neighborhood of $t$ and is a continuous point
(3) is because $\frac{\beta_i}{b_i} t + \frac{\alpha_i - a_i}{b_i} \to t$ and $F_X$ is continuous at $t$

Many of you used Slutsky Thm, which is also fine. Pretend/think a sequence of constant($\frac{\beta_i}{b_i}$ or $\frac{\alpha_i - a_i}{b_i}$) as a sequence of r.v will make life much easier in this case.

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**Q7. A&L 9.7**

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$\forall \epsilon > 0$ and $\delta > 0$
$\{X_n\}$ is stochastically bdd $\Rightarrow \forall \delta > 0, \exists M$ s.t. $sup_{n>1} P(|X_n| > M) < \frac{\epsilon}{2}$

$\{Y_n\} \to_p 0 \Rightarrow \exists n_0$ s.t. $\forall n > n_0$ $P(|Y_n| > \frac{\delta}{M}) < \frac{\epsilon}{2}$

$$\begin{aligned} P(|X_nY_n| > \delta) &= P(|X_nY_n| > \delta, |X_n| > M) + P(|X_nY_n|) > \delta, |X_n| <= M) \\ &= P(|X_nY_n| > \delta, |X_n| > M) + P(|X_nY_n|) > \delta, |X_n| <= M, |Y_n| > \frac{\delta}{M}) \\ &\leq P(|X_n| > M) + P(|Y_n| > \frac{\delta}{M}) \\ &< \epsilon \end{aligned}$$

so $X_nY_n \to_p 0$

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**Q9.**

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Assume the slow varying part will become a constant near the zero, absorb it into c.

$$\begin{aligned} P(n^{1/\alpha}(\lambda - M_n) > x) &= P(M_n < \lambda - x/n^{1/\alpha}) \\ &= P^n(X_1 < \lambda - x/n^{1/\alpha}) \\ &= (1 - c(x/n^{1/\alpha})^\alpha)^n \\ &= (1 - cx^\alpha/n)^n \\ &\to exp(-cx^\alpha) \end{aligned}$$

The guys from the third floor will tell you this is a Weibull($\alpha, c$) distribution.

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**Q10. Rosenthal 11.5.3**

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Yes, $\{\delta_i\}$ is tight. $\{x_i\}$ is compact, i.e. $\exists \{x_{ij}\}$ converge, say to $x$, then $\{\delta_{ij}\} \to_d \delta_x$, the point mass at $x$.

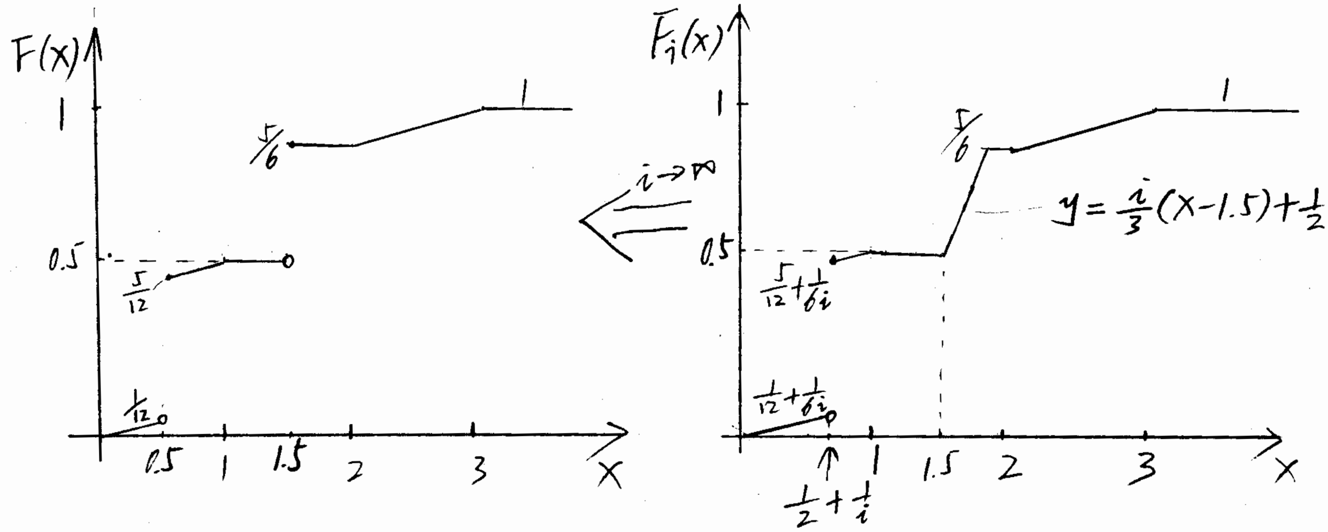


Fig 1 for Q 1

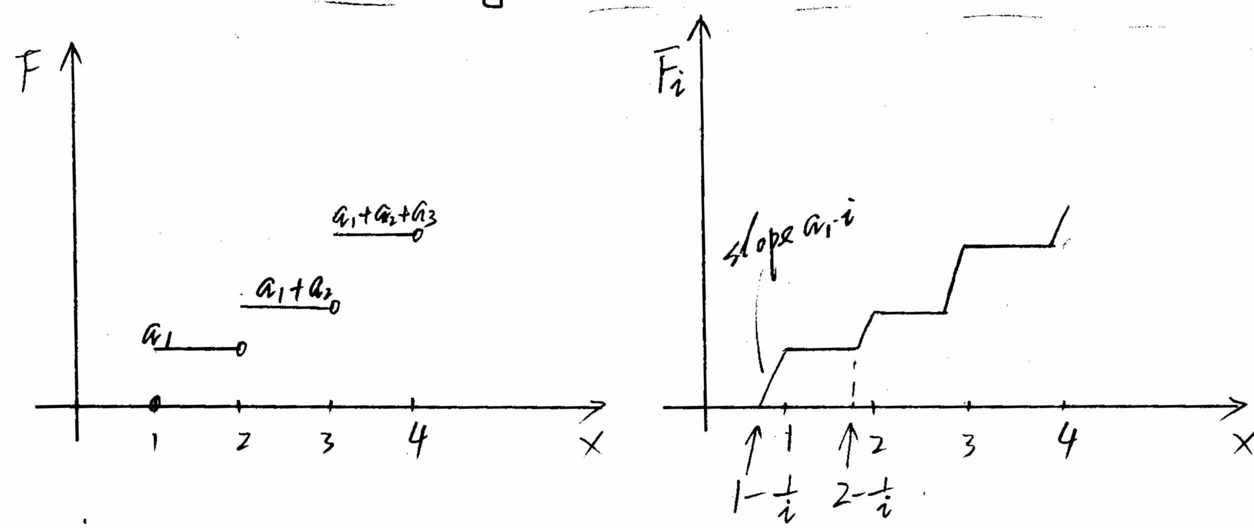


Fig 2 for Q3