I have neither given nor received unauthorized assistance on this examination.

_______________________________
Signature

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Name Printed

There are 12 parts on this exam. I will score every part out of 10 points and take your best 10 of 12 scores. (Budget your time accordingly.)
1. Consider the data model for an observable $X$ taking values in $(0,1)$ with probability density

$$f(x|\theta) = \begin{cases} 
(\theta + 1)x^\theta & \text{for } 0 < x < 1 \\
0 & \text{otherwise}
\end{cases}$$

($\theta > 0$)

a) Derive the Jeffreys prior for this model. (Remember that one form of the Fisher information in a 1-parameter model is $-E_\theta \frac{d^2}{d\theta^2} \ln f(X|\theta)$. ) Is this prior proper?

$$\ln f(x|\theta) = \ln (\theta + 1) + \theta \ln x$$

$$\frac{d}{d\theta} = \frac{1}{\theta + 1} + \ln x$$

$$\frac{d^2}{d\theta^2} = -\frac{1}{(\theta + 1)^2}$$

So $I(\theta) = \frac{1}{(\theta + 1)^2}$ and

$$J(\theta) = \sqrt{I(\theta)} = \frac{1}{\sqrt{\theta + 1}}$$

$$\int_0^\infty \frac{1}{\theta + 1} d\theta = \int_1^\infty \frac{1}{t} dt \quad \text{which diverges}$$

is not proper

b) If the posterior distribution for $\theta$ is proper, find a 95% HPD credible region for $\theta$. If it is not proper, show that it is not proper. (Remember that $x^\theta = \exp(\theta \ln x)$.)

$$\int_0^\infty \frac{1}{\theta + 1} \exp(\theta \ln x) d\theta = \frac{1}{\ln x} \exp(\theta \ln x) \bigg|_0^\infty = \frac{1}{\ln x} (0 - 1) = -\frac{1}{\ln x}$$

so the posterior is proper. Since $\ln x < 0$, $\exp(\theta \ln x)$ is decreasing in $\theta$ and an HPD credible region will be an interval of the form $(0, c)$ where

$$S = \int_0^c \exp(\theta \ln x) d\theta / \int_0^\infty \exp(\theta \ln x) d\theta$$

$$= \frac{1}{\ln x} (\exp c \ln x - 1) / \frac{1}{\ln x} (0 - 1) = 1 - \exp(c \ln x)$$

i.e. $\exp c \ln x = S$ i.e. $c \ln x = \ln S$ and

$$c = \frac{\ln S}{\ln x} = \log_x(S)$$
2. This problem concerns a simple discrete observable, $Y$, and several issues of Bayes analysis for its mean. We'll entertain two basic models with conditional distributions of $Y$ given a parameter ($\theta$ or $\gamma$) as below.

| $y$ | $f(y|\theta=1)$ | $f(y|\theta=2)$ | $f(y|\theta=3)$ | $f(y|\gamma=1)$ | $f(y|\gamma=2)$ |
|-----|------------------|------------------|------------------|------------------|------------------|
| 3   | .1               | .2               | .3               | .1               | .8               |
| 2   | .1               | .2               | .4               | .1               | .1               |
| 1   | .8               | .6               | .3               | .8               | .1               |

Consider first an analysis based on Model 1 alone.

a) Define the parametric function $\mu_i(\theta) = \mathbb{E}(Y|\theta)$. Find the 3 values $\mu_1(1), \mu_1(2)$, and $\mu_1(3)$.

\[
\begin{align*}
\mu_1(1) &= 1(.8) + 2(.1) + 3(.1) = 1.3 \\
\mu_1(2) &= 1(.6) + 2(.2) + 3(.2) = 1.6 \\
\mu_1(3) &= 1(.3) + 2(.4) + 3(.3) = 2.0
\end{align*}
\]

Suppose that I use a prior distribution for $\theta$ that is uniform on $\{1, 2, 3\}$.

b) Suppose that I observe $Y = 3$. What is the value of the posterior mean of $\mu_1(\theta)$ (given this observation)?

The posterior for $\theta$ is

\[
\begin{array}{ccc}
\theta & g(\theta|3) \\
1 & .1/(1+.2+.3) = \frac{1}{6} \\
2 & .2/(1+.2+.3) = \frac{1}{3} \\
3 & .3/(1+.2+.3) = \frac{1}{2}
\end{array}
\]

So the posterior mean for $\mu_1(\theta)$ given $Y = 3$ is

\[
\frac{1}{6}(1.3) + \frac{1}{3}(1.6) + \frac{1}{2}(2.0) = \frac{1}{6}(1.3 + 3.2 + 6.0) = \frac{1}{6}(10.5) = 1.75
\]
c) Find the value of "DIC" for this model (Model 1 with a uniform prior on $\theta$) if $Y = 3$ is observed. (Use $D_y(y)$ in place of its MCMC-based estimate $\hat{D}_y(y)$.)

$$D_A(3) = \int_{-\infty}^{\infty} -2 \ln L(\theta) d\theta |_{\theta = 3}$$

$$= -2 \left( \frac{1}{6} \ln(1) + \frac{2}{6} \ln(2) + \frac{3}{6} \ln(3) \right)$$

$$D(3) = D(3, \hat{\theta}_{MLE}(3)) = -2 \ln(3)$$

Then

$$\text{DIC} = 2D_A(3) - D(3)$$

$$= -2 \left[ \frac{1}{6} \ln(1) + \frac{2}{6} \ln(2) + \ln(3) - \ln(3) \right]$$

$$= -2 \left[ \frac{1}{3} \ln(1) + \frac{2}{3} \ln(2) \right]$$

$$= 3.68$$

Now consider both original data models, Model 1 and Model 2. For a prior on $\theta$ in Model 1, we'll continue to use a uniform prior. And now we'll also use a uniform prior for $\gamma$ on $\{1, 2\}$.

\[ a \text{ Bayes Factor} \]

d) Find the value of $A$ for comparing these two Bayes models (data model plus prior) if $Y = 3$ is observed.

$$f_1(3) = \frac{1}{3} \left( \frac{1}{3} + 2 \times 3 \right) = .2$$

$$f_2(3) = \frac{1}{2} \left( \frac{1}{2} + .3 \right) = .45$$

and

$$\frac{.45}{.2} = 2.25 \text{ for model 2 against model 1} \quad \text{(or} \quad \frac{.2}{.45} = .444 \text{ for model 1 against model 2)}$$
Parallel to what we did in Model 1, define a parametric function in Model 2 by \( \mu_2(\gamma) = \mathbb{E}(Y \mid \gamma) \).

e) (Bayesian Model Averaging) Consider now the possibility of using both models (perhaps because we're unsure about which is most appropriate). Let \( M \) be a model index (1 or 2). We'll suppose that what is of interest is the mean of \( Y \) regardless of which model is acting. That is, consider

\[
\phi = I[M = 1] \mu_1(\theta) + I[M = 2] \mu_2(\gamma)
\]

(\( I[\text{statement}] \) is 1 or 0 depending upon whether "statement" is respectively true or false.)

If I use a uniform prior on \( \{1,2\} \) for \( M \), what is the posterior mean of \( \phi \) given that I observe \( Y = 3 \)?

Note that \( \mu_2(1) = .8 + .1(2) + .1(3) = 1.3 \) and \( \mu_2(2) = .1 + .1(2) + .8(3) \)

\[
= 2.7
\]

and the posterior mean for \( \phi \) given \( M = 2, Y = 3 \) is

\[
\frac{1}{.1+.8} (1.3) + \frac{.8}{.1+.8} (2.7) = \frac{1}{3} (1.3 + .8(2.7)) = \frac{1}{3} (22.3)
\]

\[
= 2.54
\]

The joint probabilities for \( M \) and \( Y \) are such that

\[
P[M = 1 \text{ and } Y = 3] = \frac{1}{2} \left[ \frac{1}{3} \left[ .1 + .2 + .3 \right] \right] = \frac{1}{10} = .1
\]

\[
P[M = 2 \text{ and } Y = 3] = \frac{1}{2} \left[ \frac{1}{2} \left[ .1 + .8 \right] \right] = .225
\]

So \( P[M = 2 / Y = 3] = \frac{.225}{.225 + .1} = \frac{225}{325} = .6923 \)

and the posterior mean of \( \phi \) given \( Y = 3 \) is thus

\[
(1 - .6923)(1.75) + .6923(2.54)
\]

\[
= 2.30
\]
3. (Truncation) Consider again Model 1 from the previous problem, but now add the issue of truncation. That is, suppose that there is a parameter \( \tau \in \{1, 2, 3\} \) and a two-parameter model for \( Y \) with probability mass functions

\[
f(y|\theta, \tau) = \begin{cases} \frac{f(y|\theta)}{\sum_{i=1}^{\tau} f(i|\theta)} & \text{if } y \leq \tau \\ 0 & \text{otherwise} \end{cases}
\]

\( f(y|\theta, \tau = 3) = f(y|\theta) \) and truncation has no effect for \( \tau = 3 \). If \( \tau = 1 \) the distribution of \( Y \) is concentrated on \( Y = 1 \). And if \( \tau = 2 \) there is no possibility that \( Y = 3 \) and for each \( \theta \) one divides the tabled probabilities for \( Y = 1 \) and \( Y = 2 \) on page 3 by their sum to make distributions over \( \{1, 2\} \).

Suppose that \textit{a priori} \( \theta \) and \( \tau \) are independent and both are uniform on \( \{1, 2, 3\} \). If \( Y = 2 \) is observed, what is the marginal posterior distribution of \( \tau \)?

\[
\begin{align*}
&\frac{1}{1} \quad f(y|\tau=1, \theta=1) \\
&\frac{1}{2} \quad f(y|\tau=2, \theta=2) \\
&\frac{1}{3} \quad f(y|\tau=3, \theta=3)
\end{align*}
\]

So the probability \( Y = 2 \) is \( \frac{1}{3} \left[ \frac{1}{9} + \frac{1}{4} + \frac{4}{7} + 1 + 2 + 4 \right] \)

and the joint probability \( Y = 2 \) and \( \tau = 2 \) is

\[ \frac{1}{9} \left[ \frac{1}{9} + \frac{1}{4} + \frac{4}{7} \right] \]

So the posterior probability \( \tau = 2 \) is \( \frac{1}{9} + \frac{1}{4} + \frac{4}{7} \)

and the posterior probability \( \tau = 3 \)

is \( \frac{1}{9} + \frac{1}{4} + \frac{4}{7} + \frac{7}{7} = \frac{4288}{37} \)
4. In a **BUGS** simulation where some joint distribution for \( \mathbf{X} = (X_1, X_2, \ldots, X_k) \) has been specified, the software will not allow a user to define a deterministic function \( g(\mathbf{X}) \) and then specify as "data" a value for that function, unless \( g(\mathbf{X}) \) is a coordinate function (i.e. \( X_1, X_2, \ldots \) or \( X_k \)). For example, one may **not** make the assignment

\[
Y \leftarrow X_1 + X_2
\]

in the model specification, and then subsequently use data like

\[
\text{list}(Y=3)
\]

Presumably the reason for this is that while it's clear how to use \( f(3, x_2, x_3, \ldots, x_k) \) as a conditional density for \( (X_2, X_3, \ldots, X_k) \) given \( X_1 = 3 \), it's not even completely obvious in general how to think about a density on the part of \( k \)-space where \( g(\mathbf{x}) = 3 \). A way to possibly use data like \( g(\mathbf{x}) = 3 \) in **BUGS** might be to suppose that \( g(\mathbf{x}) \) plus a small random error is known to be 3.

a) Use this idea and write **BUGS** code to find (approximately) the conditional distributions of \( X_1, X_2, \) and \( \alpha \) given \( X_1 + X_2 = 1 \) in a model where

1. given \( \alpha \), \( X_1 \) and \( X_2 \) are independent, \( X_1 \sim \text{Beta}(\alpha, 1) \) and \( X_2 \sim \text{Beta}(1, \alpha) \), and
2. \( \alpha \sim \text{Exp}(1) \).

(Give both the model code and the data list.)

```plaintext
model {
  x1 ~ dbeta(alpha,1)
  x2 ~ dbeta(1,alpha)
  alpha ~ dexp(1)
  mean<-x1+x2
  sum ~ dnorm(mean,1000000)
}
list(sum=1)
```
b) Vardeman ran some code like you are supposed to write for part a) and got results below. What do the plots indicate about the MCMC in OpenBUGS? (Do you like the plots? If not, what do they indicate will be needed in order to produce reliable results based on the simulations?) Why (in retrospect) is what you see on the plots not surprising?

These two plots are very unhappy. They indicate that the MC is moving very very slowly around the parameter space. 205701 iterations are not enough to produce a smooth approximate posterior for $X_1$ and $X_2$ as is clear from this. This is not terribly surprising since the model element that says $X_1 + X_2 \sim \text{Normal with mean 1 and std deviation 0.001}$ means that with near certainty $1 - 0.004 < X_1 + X_2 < 1 + 0.004$, i.e. that $X_1 + X_2$ falls in the shaded region below. That means that $X_1$ and $X_2$ updates cannot move very far (as they involve either horizontal or vertical moves on a plot like this).

I would take a huge MCMC run (maybe 1000 times the one represented here) in order to really detail the posterior marginals of $X_1$ and $X_2$. 
5. We consider here a Bayes analysis of some (incomplete) binomial time series data. That is, suppose that conditioned on values $p_1, p_2, \ldots, p_{20}$ the variables $X_1, X_2, \ldots, X_{20}$ are independent Binomial(50, $p_i$) variables, and we adopt a random walk model for the logits of the $p_i$ time series. That is for $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{20}$ iid $N(0, \sigma^2)$ let

\[ s_i = \sum_{j=1}^{i} \varepsilon_j \]

and for constant $\mu$ suppose that

\[ \ln \left( \frac{p_i}{1-p_i} \right) = \mu + s_i, \text{ that is, } p_i = \frac{\exp(\mu + s_i)}{1 + \exp(\mu + s_i)} \]

(For what it is worth, one might call this two-parameter model a "hidden Markov model.") To complicate things further, we'll suppose that only some of the $X_i$ are observed, namely $X_1 = 3, X_3 = 15, X_4 = 11, X_8 = 18, X_{10} = 20, X_{13} = 26, X_{15} = 21, X_{20} = 13$, and set as our goal inference for the $p_i$. Some BUGS code and resulting summaries follow this page.

a) How does $\hat{p}_{20} = (X_{20} / 50) = .26$ compare to the posterior median of $p_{20}$? Explain why this is reasonable in light of the observed values $X_i$.

The posterior median is .2827, somewhat larger than $\hat{p}_{20}$. This is completely reasonable, as the last $\hat{p}$ in the data set (before $\hat{p}_{20}$) is $\hat{p}_{15} = .42$. Thus is there evidence in the data that this "random walk on logits" model produced a $\hat{p}_{15}$ considerably above .26 and thus must drift down to produce a value for $p_{20}$. So a "prior" mean for $p_{20}$ coming from a posterior based on all other observed values would be above $\hat{p} = .26$, pulling the actual posterior median up from $\hat{p}_{20}$.

b) Which $p_i$ seems to be known least precisely? Explain how you make this judgment. Then explain why this is reasonable in light of the observed values $X_i$.

$p_7$ has the largest difference between the approximate .975 quantile and approximate .925 quantile of its posterior and in this measure is known least precisely. Thus are probably 2 effects here, 1st is the effect that index 17 is in the middle of the largest run of indices without data (16, 17, 18, 19) in the time series, 2nd is the fact that from $X_{15} = 21$ ($\hat{p} = .42$) there is an indication that $p_{17}$ might be moderate and thus even if we did have $X_{17}$ it would carry less information about $p_{17}$ than if $p_{17}$ were extreme.
model {
  logsigma~dflat()
  sigma<-exp(logsigma)
  tau<-exp(-2*logsigma)
  for (i in 1:20) {
    epsilon[i] ~ dnorm(0,tau)
  }
  for (i in 2:20) {
    sum[i]<- sum[i-1] + epsilon[i]
  }
  mu ~ dnorm (0, .0001)
  for (i in 1:20) {
    logt[i] <- mu + sum[i]
  }
  for (i in 1:20) {
    p[i] <- exp(logt[i])/(1 + exp(logt[i]))
  }
  for (i in 1:20) {
    X[i] ~ dbin(p[i],50)
  }
}

list(X=c(3,NA,15,11,NA,NA,NA,18,NA,20,NA,NA,26,NA,21,NA,NA,NA,NA,NA,NA,13))
list(epsilon=c(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0),mu=0,logsigma=1)
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<td>0.1009</td>
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<td>0.1103</td>
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</tr>
</tbody>
</table>

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**Note:** Differences between .975 and .925 quantiles for the entries of `p`.