Stat 543 4-6-05

Note Title 4/6/2005

Recall

Suppose $\phi_{\theta_0}(x)$ is a size α test of $H_0: \theta = \theta_0$

Define $S(x) = \{ \theta \in \Theta \mid \phi_\theta(x) < 1 \}$

This is a confidence procedure for θ with confidence level $\geq 1-\alpha$

$$P_\theta[\theta \in S(X)] = P_\theta[\phi_\theta(X) < 1]$$
$$= E_\theta I[1 - \phi_\theta(X) > 0]$$

OOOOPS ... THIS FIRST INEQUALITY IS BACKWARDS!!

SORRY

SBV

$$\leq E_\theta\left(1 - \phi_\theta(X)\right)$$

$$= 1 - \alpha$$

Complementary Fact: confidence procedures can be used to carry out tests – Suppose $S(x)$ is a set estimator of θ with

$$P_\theta\left[\theta \in S(X)\right] \geq \eta$$

Then define $\phi_\theta(x) = I\left[\theta \notin S(x)\right]$ — I

claim that $\phi_{\theta_0}(x)$ is level $\leq 1-\eta$ for $H_0: \theta = \theta_0$

$$\pi_{\phi_\theta}(\theta) = P_\theta\left[\phi_\theta(X) = 1\right] = P_\theta\left[\theta \notin S(X)\right] \leq 1-\eta$$

ϕ_θ nonrandomized

<u>Example</u> $X_1, X_2, \ldots, X_n$ iid $N(\mu, 1)$

UMP size $\alpha = .05$ tests of $H_0: \mu = \mu_0$ vs $H_a: \mu > \mu_0$ are of the form

$$\phi_{\mu_0}(x) = \begin{cases} 1 & \text{if } \bar{x} \geq \mu_0 + 1.645 \frac{1}{\sqrt{n}} \\ 0 & \text{otherwise} \end{cases}$$

and the corresponding 95% level confidence sets are

$$S(x) = \left\{ \mu \,\middle|\, \bar{x} \leq \mu + 1.645 \frac{1}{\sqrt{n}} \right\}$$

$$= \left\{ \mu \,\middle|\, \mu \geq \bar{x} - 1.645 \frac{1}{\sqrt{n}} \right\}$$

$$= \left[\bar{x} - 1.645 \frac{1}{\sqrt{n}}, \infty \right)$$

This business of inverting tests of point null hypotheses to get confidence procedures is perfectly general — and when the tests used to make $S(x)$ have optimality properties, $S(x)$ inherits those

Thm Suppose that for each $\theta_0 \in \Theta$, $\Delta(\theta_0) \subset \Theta$ with $\theta_0 \notin \Delta(\theta_0)$ and ϕ_{θ_0} is a nonrandomized UMP test of size α for $H_0: \theta = \theta_0$ vs $H_a: \theta \in \Delta(\theta_0)$ Then if $S(x)$ is the confidence procedure derived from the tests (ϕ_θ) and $U(x)$ is any other $1-\alpha$ level confidence procedure Then

$$P_\theta\left[\theta_0 \in U(X)\right] \geq P_\theta\left[\theta_0 \in S(X)\right] \quad \forall\ \theta \in \Delta(\theta_0)$$

(i.e. $S(x)$ minimizes the probability of covering wrong values of θ)

Example $X_1, X_2, \ldots, X_n$ iid $N(\mu, 1)$

$$S(x) = \left[\bar{x} - 1.645 \frac{1}{\sqrt{n}}, \infty\right)$$

is derived from UMP tests of $H_0: \mu = \mu_0$ vs $H_a: \mu > \mu_0$

– by the Theorem for any $\mu > \mu_0$

$S(x)$ minimizes $P_\mu\left[\mu_0 \in S(X)\right]$

Pf: Define from $U(x)$ tests of $H_0: \theta = \theta_0$

$$\phi'_{\theta_0}(x) = \begin{cases} 1 & \text{if } \theta_0 \notin U(x) \\ 0 & \text{if } \theta_0 \in U(x) \end{cases}$$

Then

$$\pi_{\phi'_{\theta_0}}(\theta_0) = P_{\theta_0}\left[\theta_0 \notin U(X)\right] \le \alpha$$

since $U(X)$ is a level $1-\alpha$ confidence method

i.e. $\phi'_{\theta_0}(x)$ is a test of $H_0: \theta = \theta_0$ of size $\le \alpha$

The fact that ϕ_{θ_0} is a nonrandomized UMP size α test of $H_0: \theta = \theta_0$ vs $H_a: \theta \in \Delta(\theta_0)$ means

$$\Pi_{\phi_{\theta_0}}(\theta) \ge \Pi_{\phi'_{\theta_0}}(\theta) \qquad \forall\, \theta \in \Delta(\theta_0)$$

$$P_\theta\left[\theta_0 \notin S(X)\right] \qquad\qquad P_\theta\left[\theta_0 \notin U(X)\right]$$

Chapter 5 of B+D applies large sample theory (convergence in probability and dsn) to inference

Stat 542 $\qquad X_n \xrightarrow{\mathcal{L}} X$

Stat 543 $\qquad X_n \xrightarrow{\mathcal{L}_\theta} X \quad \forall\, \theta \in \Theta$ (or perhaps $\forall\, \theta \in \Theta' \subset \Theta$)

This can be come useful in inference

Example It's "Stat 101 fact" that

$$\frac{\hat{p}_n - p}{\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}} \text{ is approximately } N(0,1)$$

This is an approximate pivot . . . This e.g. leads to approximate confidence limits for p

$$\hat{p}_n \pm z\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}$$