Stat 543 4-20-05

Recall: Thm 9 of handout

$$2\left(\ell_n(\delta_n(X)) - \ell_n(\theta_0)\right) \xrightarrow{\mathcal{L}_{\theta_0}} \chi^2_1$$

loglikelihood at the "MLE"

loglikelihood at the "true value"

This is sort of

$$2 \ln \frac{\sup_\theta L_n(\theta)}{L_n(\theta_0)}$$

likelihood ratio statistic for testing $H_0: \theta = \theta_0$ vs $H_a: \theta \neq \theta_0$

L_n the likelihood
ℓ_n the loglikelihood

this gives a way to set cut-off values for LRT's — it also gives a way to make large sample confidence sets for θ — This amounts to the following, if c is the upper α pt of χ^2_1

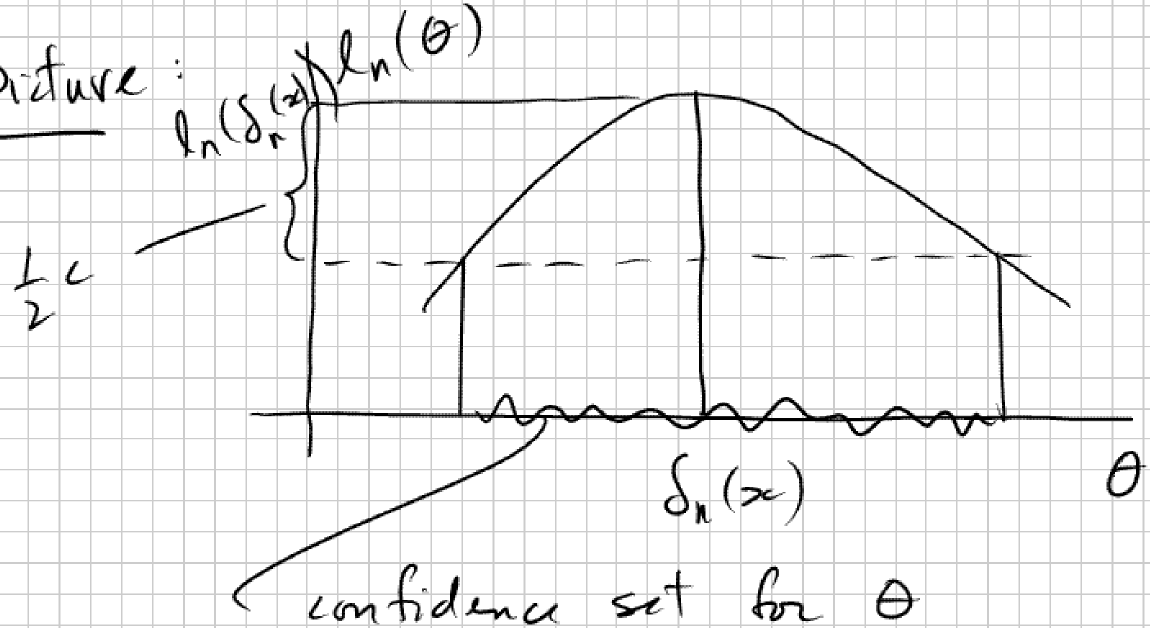
$$\underbrace{P_{\theta_0}\left[2\left(l_n(\delta_n(X)) - l_n(\theta_0)\right) < c\right]}_{P_{\theta_0}\left[l_n(\delta_n(X)) - \frac{1}{2}c < l_n(\theta_0)\right]} \approx 1-\alpha$$

So ... the set of θ's with $l_n(\theta)$ no more than $\frac{1}{2}c$ below the sup of the log-likelihood functions as a confidence set for Θ, that is with data $X = x$

$$\{ \theta \mid \ell_n(\theta) > \ell_n(\delta_n(x)) - \tfrac{1}{2}c \}$$

can be used as a confidence set for θ

Picture :



"Argument" for Thm 9 of handout — write $\hat{\theta}_n$ in place of $\delta_n(X)$

Basic Idea: Expand $l_n(\cdot)$ in a Taylor Series around $\hat{\theta}_n$

Note BTW that the argument for asymptotic normality of $\hat{\theta}_n$ is done by expanding $l_n'(\cdot)$ in a Taylor Series around θ_0

$$l_n(\theta_0) = l_n(\hat{\theta}_n) + (\theta_0 - \hat{\theta}_n) l_n'(\hat{\theta}_n) + \frac{1}{2}(\theta_0 - \hat{\theta}_n)^2 l''(\hat{\theta}_n) + \frac{1}{6}(\theta_0 - \hat{\theta}_n)^3 l_n'''(\theta_1)$$

where θ_1 is between θ_0 and $\hat{\theta}_n$

So

$$2\left(\ell_n(\hat{\theta}_n) - \ell_n(\theta_0)\right) = \underbrace{-2(\theta_0 - \hat{\theta}_n)\,\ell_n'(\hat{\theta}_n)}_{A_n}$$

$$\underbrace{-2\left(\tfrac{1}{2}\right)(\theta_0 - \hat{\theta}_n)^2\,\ell_n''(\hat{\theta}_n)}_{B_n}$$

$$\underbrace{-\tfrac{1}{6}(2)\,(\theta_0 - \hat{\theta}_n)^3\,\ell_n'''(\theta_1)}_{C_n}$$

Consider first A_n — A_n is 0 because by hypothesis $\ell_n'(\hat{\theta}_n) = 0$

Next consider $B_n = (\theta_0 - \hat{\theta}_n)^2\left(-\ell_n''(\hat{\theta}_n)\right)$

$$= \underbrace{\left(\sqrt{n}(\theta_0 - \hat{\theta}_n)\right)^2}\ \underbrace{\left(-\tfrac{1}{n}\,\ell_n''(\hat{\theta}_n)\right)}$$

$$\xrightarrow{\mathcal{L}_{\theta_0}} \left(N(0, I_1^{-1}(\theta_0))\right)^2 \qquad \text{the observed FI} \xrightarrow{P_{\theta_0}} I_1(\theta_0)$$

So $\quad B_n \xrightarrow{\mathcal{L}_{\theta_0}} \left(N(0, I_1^{-1}(\theta_0)) \cdot \sqrt{I_1(\theta_0)} \right)^2$

and sure enough standard regularity conditions are set up (just as in the argument for asymptotic normality of $\hat{\theta}_n$) to get me $\quad C_n \xrightarrow{P_{\theta_0}} 0$

There is an important multivariate version of χ^2 limit for the LRT test statistic — that goes

$$\underset{k\times 1}{\theta} = \begin{pmatrix} \underset{l\times 1}{\theta_1} \\ \underset{(k-l)\times 1}{\theta_2} \end{pmatrix}$$

$\hat{\theta}_n$ an "MLE" similarly partitioned

Suppose that for each $\theta_1 \in \Re^l$

$\theta^*_{n2}(\theta_1) \in \Re^{k-l}$ is a "maximizer" of $l_n(\theta_1, \cdot)$ over choices of θ_2

Let

$$l_n^*(\theta_1) = l_n(\theta_1, \theta^*_{n2}(\theta_1)) = \max_{\theta_2} l_n(\theta_1, \theta_2)$$

is called the "profile log-likelihood" for θ_1

and this can be treated as if it were a log likelihood for θ_1 to do inference for θ_1

"Thm" Under appropriate regularity conditions in an iid model

$$2\left(\ell_n(\hat{\theta}_n) - \ell_n^*(\theta_{01})\right) \xrightarrow{\mathcal{L}_{\theta_0}} \chi^2_l$$

max of $\ell_n(\cdot)$ is also max of $\ell_n^*(\cdot)$

is sort of $\quad 2 \ln \dfrac{\max L_n(\theta)}{\max_{\theta \text{ with } \theta_1 = \theta_{01}} L_n(\theta)}$