Stat 543 4-15-05

Recall: $\delta_n(X)$ an "MLE"

$$\sqrt{n}\left(\delta_n(X) - \theta_0\right) \xrightarrow{\mathcal{L}_{\theta_0}} N\left(0, \frac{1}{I_1(\theta_0)}\right)$$

$\Rightarrow$ (unusable) confidence limits for θ

$$\delta_n(X) \pm z \frac{1}{\sqrt{n I_1(\theta)}}$$

☹ $I(\theta)$ usually depends on θ

?? how to modify / extend This to provide something that's usable ?

Corollary 7 of handout under appropriate regularity conditions ($\delta_n(X)$ as before)

$$\underbrace{\sqrt{I_1(\delta_n(X))}\sqrt{n}\left(\delta_n(X) - \theta_0\right)}_{} \xrightarrow{\mathcal{L}_{\theta_0}} N(0,1)$$

Why? above is

$$\sqrt{\frac{I_1(\delta_n(X))}{I_1(\theta_0)}} \quad \sqrt{n I_1(\theta_0)}\left(\delta_n(X) - \theta_0\right)$$

$$\sqrt{\frac{I_1(\delta_n(X))}{I_1(\theta_0)}} \xrightarrow{P_{\theta_0}} 1$$

provided $I_1(\cdot)$ is cont$\underline{s}$

$$\sqrt{n I_1(\theta_0)}\left(\delta_n(X) - \theta_0\right) \xrightarrow{\mathcal{L}_{\theta_0}} N(0,1)$$

so the product converges in dsn to $N(0,1)$

<u>Example</u> $X_1, X_2, \ldots, X_n$ iid Bernoulli p

$$I_1(p) = \frac{1}{p(1-p)}$$

so replacing unusable confidence limits

$$\hat{p} \pm z\sqrt{\frac{1}{n I_1(p)}} \quad \text{i.e. } \hat{p} \pm z\sqrt{\frac{p(1-p)}{n}}$$

<u>by</u> $$\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

This route to "fixing" the unusable confidence limits is sometimes called using the "Expected Fisher information"

A second route is to make use of the "sample Fisher information" — Note that

$$I(\theta_0) = nI_1(\theta_0) = n\left(E_{\theta_0} - \frac{\partial^2}{\partial\theta^2} \log f(X_1 \mid \theta) \Big|_{\theta=\theta_0} \right)$$

a mean that can perhaps be approximated by a sample average

$$\frac{1}{n} l_n''(\theta_0) = \frac{1}{n} \sum_{i=1}^{n} \frac{d^2}{d\theta^2} \log f(X_i \mid \theta) \Big|_{\theta=\theta_0}$$

so by WLLN $\quad \frac{1}{n} l_n''(\theta_0) \xrightarrow{P_{\theta_0}} -I_1(\theta_0)$

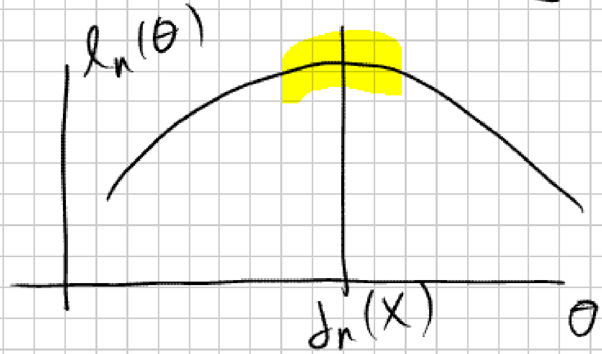
and if things are nice $\quad \frac{1}{n} l_n''(\theta_0) - \frac{1}{n} l_n''(\delta_n(X))$

goes to 0 in θ_0 probability ... and hopefully this makes plausible

Corollary 8 on handout under appropriate conditions

$$\sqrt{-\ell_n''(\delta_n(X))}\,\left(\delta_n(X) - \theta_0\right) \xrightarrow{\mathcal{L}_{\theta_0}} N(0,1)$$

(I've replaced the FI evaluated at $\delta_n(X)$ with minus the curvature of loglikelihood evaluated at $\delta_n(X)$)



And this leads to confidence limits for θ

$$\delta_n(X) \pm z \frac{1}{\sqrt{-\ell_n''(\delta_n(X))}}$$

(this is called use of the "observed FI")

Example   iid Bernoulli

$$\ell_n(p) = \Sigma X_i \ln p + (n - \Sigma X_i) \ln(1-p)$$

$$\ell_n'(p) = \frac{\Sigma X_i}{p} - \frac{n - \Sigma X}{1-p}$$

$$\ell_n''(p) = -\frac{\Sigma X_i}{p^2} - \frac{n - \Sigma X_i}{(1-p)^2}$$

$$l_n''(\hat{p}) = -\frac{\Sigma X_i}{\left(\frac{\Sigma X_i}{n}\right)^2} - \frac{n - \Sigma X_i}{\left(\frac{n - \Sigma X_i}{n}\right)^2}$$

$$= -n\left(\frac{1}{\hat{p}} + \frac{1}{1-\hat{p}}\right)$$

$$= \frac{-n}{\hat{p}(1-\hat{p})}$$

So we get approximate confidence limits

$$\hat{p} \pm z \frac{1}{\sqrt{-l_n''(\hat{p})}}$$

i.e. $$\hat{p} \pm z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

!!! I get the same interval as when I used "Expected FI^" — that always happens in exponential families — in other cases there can be a difference ... when there is the 2nd of these (according to folklore) does a better job of holding nominal coverage levels (for moderate n) than the 1st —

These results (Corollary 7 + 8) cover estimation of a one dimensional parameter -- what if θ is $k \times 1$ and I wish to do confidence set estimation of all or part of θ) — I can use the approximate MVN dsn of an "MLE" —

How? Use a Stat 542 dsn fact

$$Y \sim N_k(\mu, \Sigma)$$

$$(Y-\mu)' \Sigma^{-1} (Y-\mu) \sim \chi^2_k$$

How? if c is the upper α pt of χ^2_k dsn

$$P\left[(Y-\mu)' \Sigma^{-1} (Y-\mu) < c \right] = 1-\alpha$$

and $\therefore$ if $Y = y$

$$\left\{ \mu \in \mathbb{R}^k \;\middle|\; (y-\mu)' \Sigma^{-1} (y-\mu) < c \right\}$$

is a $(1-\alpha) \times 100\%$ confidence set for μ