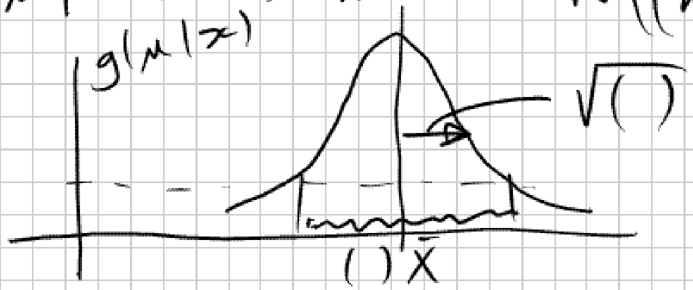
Stat 543 4-1-05

Recall Bayes set estimation + prediction ...
HPD sets (HPD "credible" sets)

Example $X_1, X_2, \ldots, X_n, X_{n+1}$ iid $N(\mu, 1)$

$\mu \sim N(0, \gamma^2)$ credible sets for μ, X_{n+1} based on observables? $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$\mu | X_1, \ldots, X_n$ is $N\left(\left(\frac{n\gamma^2}{n\gamma^2+1}\right)\bar{X}, (\quad)\right)$



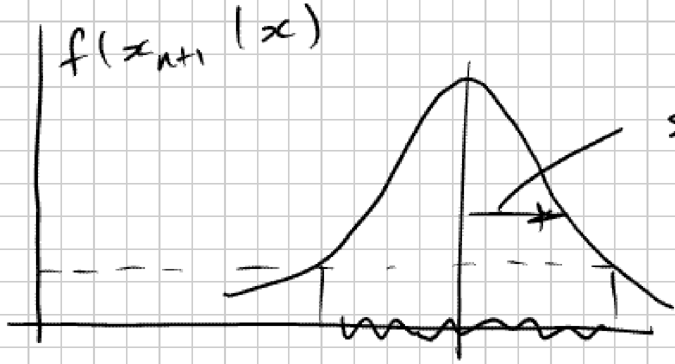
from picture, a 95% HPD credible interval for μ has endpoints

$$(\quad)\bar{X} \pm 1.96\sqrt{(\quad)}$$

to make a credible interval for X_{n+1} (based on $X_1, \ldots, X_n$) using the conditional dsn of X_{n+1} given $X_1, \ldots, X_n$

$X_{n+1} \mid X_1, \ldots, X_n$ is $N\left((\quad)\bar{X},\ 1+\frac{\gamma^2}{n\gamma^2+1}\right)$

from B&D problem 1.2.14 on HW #1



std dev = $\sqrt{1+\frac{\gamma^2}{n\gamma^2+1}}$

$$\bar{X}\left(\frac{n\gamma^2}{n\gamma^2+1}\right)$$

so, e.g., 95% credible limits are

$$\bar{X}\left(\frac{n\gamma^2}{n\gamma^2+1}\right) \pm 1.96\sqrt{1+\frac{\gamma^2}{n\gamma^2+1}}$$

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at least in theory, that's end of the Bayes story ... The problem, of course, is the details of calculation in less nice models ...

I should admit (particularly where some kind of simulation is used to get a sample from the posterior as a means of doing Bayes inference) that it is common to drop the "HPD" business and settle for Bayes credible intervals of the form

$$\left(\begin{matrix} \text{lower } \alpha/2 \\ \text{pt of} \\ \text{posterior} \end{matrix} , \begin{matrix} \text{upper } \alpha/2 \\ \text{pt of} \\ \text{posterior} \end{matrix} \right)$$

(this is an $1-\alpha$ level credible interval)

Classical / non-Bayesian Set Estimation & Prediction

here the "reliability" of a method is not stated in terms of a posterior probability but rather

Def The confidence level of $S(X)$ is

$$\inf_{\theta} P_\theta\left[\,\gamma(\theta) \in S(X)\,\right]$$ (for a set estimator of $\gamma(\theta)$) or

$$\inf_{\theta} P_\theta\left[\,Y \in S(X)\,\right]$$ (for a set prediction of Y)

Standard methods (Stat 500/511) are based on (exact or approximate) "pivots"/"pivotal quantities" — a pivot is a function

$$p(X, \theta) \qquad (\, p(X, Y)\,)$$

whose dsn is (at least approximately) free of θ

Examples 1-sample normal

$$\frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1} \implies$$ "t" limits for μ $\bar{X} \pm t \frac{s}{\sqrt{n}}$

a "pivot"

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1} \implies$$ χ^2 limits for σ or σ^2

$$\frac{\bar{X}_n - X_{n+1}}{S_n\sqrt{1+\frac{1}{n}}} \sim t_{n-1} \implies$$ t limits for X_{n+1} $\bar{X}_n \pm t\, S_n\sqrt{1+\frac{1}{n}}$

2 sample normal model

a pivot
$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1,\,n_2-1} \implies$$
F confidence limits for $\frac{\sigma_1}{\sigma_2}$

etc. ...

beyond the Stat 500 / Stat 511 cases, at least for exact pivots, this is a case-by-case "methodology" — the only general theory (beyond LM stuff) here is large sample theory (the stuff of Ch 5 of B+D) — coming soon

Before starting Ch 5 there is one issue regarding set estimation that can be discussed

in general terms — that is a "duality"
(connection) between testing + set estimation

<u>still</u> $X \sim f(x|\theta)$ with parameter space Θ

suppose that for each $\theta_0 \in \Theta$

$\phi_{\theta_0}(x)$ is a size α test of $H_0 : \theta = \theta_0$

Define

$$S(x) = \underbrace{\left\{ \theta \in \Theta \;\middle|\; \phi_\theta(x) < 1 \right\}}$$

the set of parameters whose associated tests have positive (conditional on x) probability of acceptance if all of the ϕ_θ are non-randomized, $S(x)$ is the set of parameters whose associated tests accept

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$S(X)$ is a confidence procedure for the estimation of θ with confidence level $\geq 1-\alpha$