Stat 543 3-9-05

"Simple vs Simple" Testing (+ The N-P Lemma)

Suppose $\Theta = \{\theta_0, \theta_1\}$ $\Theta_0 = \{\theta_0\}$ $\Theta_1 = \{\theta_1\}$

$$f(x|\theta_0) \quad \text{vs} \quad f(x|\theta_1)$$

Bayes look at This problem : $g(\theta_0) + g(\theta_1) = 1$

$$f(x,\theta) = f(x|\theta)g(\theta)$$

$$g(\theta|x) = \frac{f(x|\theta)\,g(\theta)}{f(x|\theta_0)g(\theta_0) + f(x|\theta_1)g(\theta_1)}$$

Bayes rule is "minimize posterior expected loss"

If $a=0$ posterior expected loss is

$$1 \cdot g(\theta_1|x)$$

If $a=1$ posterior expected loss is

$$1 \cdot g(\theta_0|x)$$

So "obviously" choose between $a=0$ and $a=1$ depending upon how $g(\theta_1|x)$ compares to $g(\theta_0|x)$ i.e. we choose action corresponding to the θ with largest posterior probability

So a Bayes test for this simple vs simple problem is of the form

$$\phi(x) = \begin{cases} 1 & \text{if } g(\theta_1|x) > g(\theta_0|x) \\ 0 & \qquad\qquad < \end{cases}$$

$$= \begin{cases} 1 & \text{if } f(x|\theta_1)\,g(\theta_1) > f(x|\theta_0)\,g(\theta_0) \\ 0 & \qquad\qquad < \end{cases}$$

$$= \begin{cases} 1 & \dfrac{f(x|\theta_1)}{f(x|\theta_0)} > \dfrac{g(\theta_0)}{g(\theta_1)} \\ 0 & \qquad\qquad < \end{cases}$$

i.e. good (Bayes) tests reject H_0 (decide $a=1$) for large values of the "likelihood ratio"

$$R(x) = \frac{L_x(\theta_1)}{L_x(\theta_0)}$$

Example $\quad X \sim N(\mu, 1) \qquad \mu_0 < \mu_1$

$$R(x) = \frac{\frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}(x-\mu_1)^2}{\frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}(x-\mu_0)^2}$$

$$\vdots$$

$$= \exp\left[x(\mu_1-\mu_0)\right] \exp\left[\mu_1^2 - \mu_0^2\right]$$

in x so Bayes tests reject H_0 (decide $a=1$) for large x

Example   Simple discrete

x	0	1	2	3	4	5
$f(x\|\theta_1)$	.2	.1	.4	.2	.05	.05
$f(x\|\theta_0)$	.2	.1	.2	.05	.3	.15
$R(x)$	1	1	2	4	$\frac{1}{6}$	$\frac{1}{3}$

So "good" tests look like

$$\phi_1(x) = \begin{cases} 1 & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

or

$$\phi_2(x) = \begin{cases} 1 & \text{if } x = 3 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases}$$

or $\phi_3(x) = \begin{cases} 1 & x = 3 \text{ or } 2 \text{ or } 1 \\ 0 & \end{cases}$

or

$\phi_4(x) = \begin{cases} 1 & x = 3 \text{ or } 2 \text{ or } 0 \\ 0 & \end{cases}$

or

$\phi_5(x) = \begin{cases} 1 & x = 3 \text{ or } 2 \text{ or } 0 \text{ or } 1 \\ 0 & \end{cases}$

or $\phi_6(x) = \begin{cases} 1 & x = 3 \text{ or } 2 \text{ or } 0 \text{ or } 1 \text{ or } 5 \\ 0 & x = 4 \end{cases}$

Notice that this prescription (reject H_0/decide $a=1$ when $R(x)$ is big) adds points to rejection regions by reducing what is "large" and this is in some sense buying "power" ($f(x|\theta_1)$) as cheaply (for smallest possible $f(x|\theta_0)$) as possible — that story makes plausible the fundamental result of testing Thy (Neyman-Pearson Lemma) —

A bit more notation/terminology:

In a simple vs simple problem for a test ϕ it's common to call

$$\pi_\phi(\theta_0) = P_{\theta_0}\left[\phi(X) = 1\right]$$
$$= \text{Type I error probability}$$

$= \alpha$

= "The 'size' of the test ϕ"

$\pi_\phi(\theta_1) = 1 - \underbrace{\text{Type II error probability}}_{\beta}$

= "The 'power' of the test ϕ"

<u>Theorem</u> Neyman - Pearson Lemma (Part I)

If $\phi(x) = \begin{cases} 1 & \text{if } R(x) > k \\ 0 & \phantom{\text{if } R(x)} < \end{cases}$

then ϕ is "most powerful of its size"

Conclusion: If ϕ' is another test with

$$\underbrace{\pi_{\phi'}(\theta_0)}_{\text{size of } \phi'} \le \underbrace{\pi_{\phi}(\theta_0)}_{\text{size of } \phi}$$

Then $$\underbrace{\pi_{\phi}(\theta_1)}_{\text{"power" of } \phi} \ge \underbrace{\pi_{\phi'}(\theta_1)}_{\text{"power" of } \phi'}$$