Stat 543 3-28-05

Recall: comments regarding UMP tests

1. primarily available in 1-parameter MLR families for one-sided alternatives

2. there are a very few other circumstances where very specialized arguments can identify UMP tests –

e.g. in the 2-parameter $N(\mu,\sigma^2)$ model with $H_0: \sigma^2 \le \sigma_0^2$ and $H_a: \sigma^2 > \sigma_0^2$ usual F tests are UMP of their size

But usual F tests of $H_0: \sigma^2 \ge \sigma_0^2$ $H_a: \sigma^2 < \sigma_0^2$ is not UMP

usual t tests of $H_0: \mu \ge \mu_0$ $H_a: \mu < \mu_0$ is not UMP

There is a weaker (and technically much harder) optimality theory that is available in multiparameter problems (and with 2-sided alternatives) – idea is that (for example) if we we're going to find a uniformly best test of $H_0: \mu = 0$ vs $H_a: \mu \neq 0$ in $N(\mu, 1)$ model, we must disallow tests like

$$\phi(x) = I[x > 1.645]$$

$$\text{and } \phi'(x) = I[x < -1.645]$$

and perhaps can do that on the basis that while their power functions are very good on part Θ, they are poor on another part of Θ,

<u>Def</u> A test ϕ is called "unbiased" of level α provided

$$\Pi_\phi(\theta) \le \alpha \quad \text{for } \theta \in \Theta_0$$

$$\text{and } \Pi_\phi(\theta) \ge \alpha \quad \text{for } \theta \in \Theta_1$$

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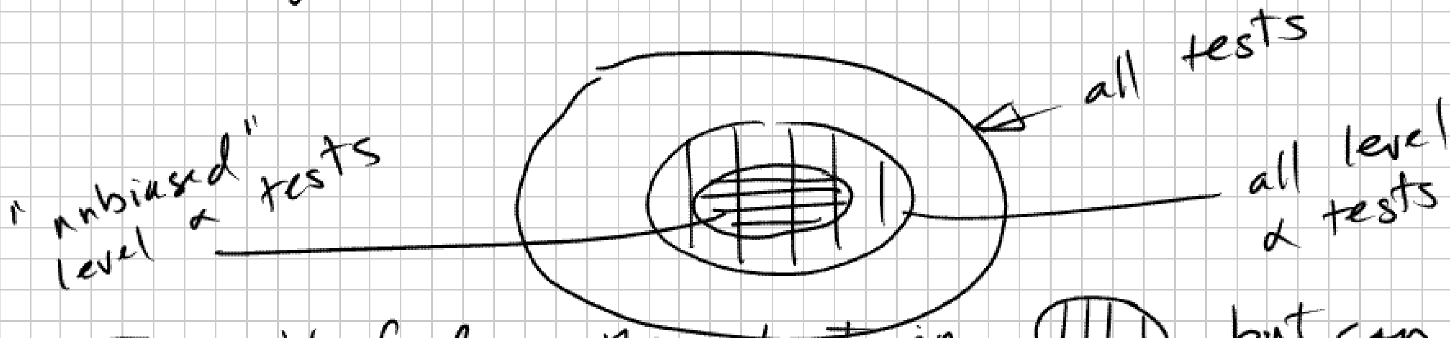
in the $N(\mu, 1)$ problem ϕ_1 ϕ' are <u>not</u> unbiased for $H_0: \mu = 0$ vs $H_a: \mu \neq 0$



I can't find anything best in (|||) but can find something best in (#)

The best test in (III) is called a UMPU test

– e.g. for testing $H_0: \mu = 0$ vs $H_a: \mu \neq 0$ in $N(\mu, 1)$ model UMPU test of size $\alpha = .05$ is

$$\phi''(x) = I\left[\,|x| > 1.96\,\right]$$

– hard to prove but there is some general UMPU theory that extends to multi-parameter exponential families — Stat 643

Instead of concentrating on optimality, now ask "What's a reasonable heuristic for test construction in general?"

Likelihood Ratio Tests

$\Theta = \Theta_0 \sqcup \Theta_1$

possibly high dimensional / complicated

Motivation

Simple vs Simple   N-P says that optimal tests reject for large

$$R(x) = \frac{f(x|\theta_1)}{f(x|\theta_0)}$$

Bayes   Bayes optimal tests (0-1 loss) reject for large

$$B(x) = \frac{\int_{\Theta_1} f(x|\theta)\, g(\theta)\, d\theta}{\int_{\Theta_0} f(x|\theta)\, g(\theta)\, d\theta}$$

or equivalently for large

$$B'(x) = \frac{\int_{\Theta_1} f(x|\theta) \frac{g(\theta)}{P_g[\theta \in \Theta_1]}\, d\theta}{\int_{\Theta_0} f(x|\theta) \frac{g(\theta)}{P_g[\theta \in \Theta_0]}\, d\theta}$$

$$= \frac{\text{θ average of $f(x|\theta)$ according to the prior dsn conditioned on $\theta \in \Theta_1$}}{\text{θ average of $f(x|\theta)$ according to the prior dsn conditioned on $\theta \in \Theta_0$}}$$

"LRT"'s reject H_0 for large (larger than k)

$$\lambda(x) = \frac{\sup_{\theta \in \Theta_1} f(x|\theta)}{\sup_{\theta \in \Theta_0} f(x|\theta)}$$

when $k > 1$ this is equivalent to rejecting for large

$$\lambda'(x) = \frac{\sup_{\theta \in \Theta} f(x|\theta)}{\sup_{\theta \in \Theta_0} f(x|\theta)} = \max\left(1, \lambda(x)\right)$$

Many common test procedures turn out to be LRT's

<u>Example</u> $X_1, X_2, \ldots, X_n$ iid $N(\mu, \sigma^2)$

$$H_0: \mu = 17 \quad \text{vs} \quad H_a: \mu \neq 17$$

$$\lambda'(x) = \frac{\sup_{\mu,\sigma^2} f(x|\mu,\sigma^2)}{\sup_{\sigma^2} f(x|17,\sigma^2)}$$

$$f(x|\mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp -\frac{1}{2\sigma^2}\sum (x_i-\mu)^2$$

MLE's of μ, σ^2 are $\hat{\mu} = \bar{x}$, $\widehat{\sigma^2} = \frac{n-1}{n} s^2$

and the maximizer (over σ^2) of $f(x|17,\sigma^2)$

$$\tilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - 17)^2$$

and Thus

$$\lambda'(x) = \frac{f(x \mid \hat{\mu}, \hat{\sigma}^2)}{f(x \mid 17, \tilde{\sigma}^2)}$$

$$= \left(\frac{\sum (x_i - 17)^2}{\sum (x - \bar{x})^2} \right)^{n/2} \qquad \frac{\overbrace{\exp -\frac{1}{2\hat{\sigma}^2} \sum (x_i - \hat{\mu})^2}^{\exp -\frac{n}{2}}}{\underbrace{\exp -\frac{1}{2\tilde{\sigma}^2} \sum (x_i - 17)^2}_{\exp -\frac{n}{2}}}$$